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Dimension of C^1-splines on type-6 tetrahedral partitions

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Abstract

We consider a linear space of piecewise polynomials in three variables which are globally smooth, i.e. trivariate C^1-splines of arbitrary polynomial degree. The splines are defined on type-6 tetrahedral partitions, which are natural generalizations of the four-directional mesh. By using Bernstein–Bézier techniques, we analyze the structure of the spaces and establish formulae for the dimension of the smooth splines on such uniform type partitions.

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1. Introduction

Spline spaces are of particular interest in approximation theory and computer aided geometric design. For splines in one variable there exists an almost completely developed

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theory (cf. [6,17,22,24]). On the other hand, much less is known for bivariate and trivariate splines (cf. [9,32], and the references therein), i.e. splines which are defined on triangulations and tetrahedral partitions, respectively. The main reason for this is that these spaces have a more complex structure than univariate spline spaces, and even the most basic problems for these spaces are sometimes difficult to solve.

Efficient approximation and interpolation methods using multivariate splines (cf. [20], and the references therein, and for instance, the bivariate approaches of [11,13,19,21]) require some knowledge on the structure of these spaces. One such basic structural question in multivariate spline theory is to determine the dimension (i.e. the number of degrees of freedom) of the spaces. This problem is easy to solve for continuous multivariate splines, but the situation is completely different and stands in striking contrast to univariate theory if we consider multivariate splines satisfying smoothness conditions.

In this case, the problem of determining the dimension of splines on given partitions becomes a complex task particularly when the degree of the splines is low. For bivariate splines on given triangulations the most general results are lower and upper bounds on the dimension (cf. [25,26]). Moreover, the dimension is known for splines on uniform partitions (cf. [10]), on arbitrary triangular cells (cf. [27]), and for certain degrees (cf. [2,14,15]). For smooth trivariate splines (non-trivial) bounds on the dimension of the spaces are difficult to obtain in general, and it has been recognized that even for splines defined on arbitrary tetrahedral (half) cells an exact dimension count would require at least some knowledge on the dimension of bivariate spline spaces of arbitrary degree (cf. [3, Example 25]; [4, Remark 66]).

There are only a few papers on the dimension of trivariate splines and in fact very little is known about these spaces to date. Early results known from the finite element literature (cf. [33]) deal with certain subspaces (which are now called super spline spaces) of splines with relatively high degree. For splines of low degrees, results are known mainly for C^1-splines. For instance, Alfeld [1] developed a local Hermite interpolation method using trivariate quintic super splines on tetrahedral partitions, where all the tetrahedra are split into four subtetrahedra (trivariate Clough–Tocher split). Quintic C^1-splines with super smoothness conditions on uniform type partitions and on certain classes of tetrahedral partitions were investigated in connection with the local interpolation methods of Schumaker and Sorokina [28], and Lai and Le Méhauté [16]. Farin and Worsey [31] generalized the bivariate Clough–Tocher element for cubic C^1-splines by splitting each tetrahedron into 12 subtetrahedra. For an application of this method in the context of so-called A-patches, see Bajaj et al. [5]. As a byproduct of these methods the dimension of the spaces on the resulting tetrahedral partitions was determined.

In this paper, we determine the dimension of trivariate C^1-splines of arbitrary polynomial degree on uniform type tetrahedral partitions Δ, where no tetrahedron is split. The partitions Δ are obtained as a natural generalization of the four-directional mesh known from the bivariate spline theory. Roughly speaking, given a uniform cube partition of a three-dimensional domain, each cube Q is subdivided into 24 tetrahedra which have the center of Q (i.e. the intersection point of the four diagonals in Q) as a common vertex (see Fig. 2, left). The partitions Δ are called type-6 tetrahedral partitions because they are obtained from slicing each cube Q with the six planes which contain two opposite edges in Q. We analyze the structure of the C^1-splines on Δ by using the piecewise Bernstein–Bézier representation of the splines and determine the dimension of the trivariate splines by constructing a suitable minimal determining set $\mathcal{M}$ for the spaces (i.e. roughly speaking a subset of the domain points such that the associated Bernstein–Bézier coefficients uniquely determine the splines while all the smoothness conditions are satisfied, see Alfeld et al. [2]). To do this, we use a well-known result (cf. [7,12], see also [9]) which characterizes C^1-smoothness across the common triangular faces of two neighboring polynomial pieces in Bernstein–Bézier representation. Our approach works as follows. We first give minimal determining sets for C^1-splines on a particular tetrahedral cell, i.e. one cube which is subdivided into 24 tetrahedra. Then, we construct step by step a minimal determining set $\mathcal{M}$ for the whole C^1-spline space. This is done inductively by considering the tetrahedra of the partition in an appropriate order (see the proof of Theorem 6.1 in Section 6), where in each step the remaining degrees of freedom are determined. Counting the number of points in $\mathcal{M}$ we obtain explicit formulae for the dimension of the C^1-spline spaces of arbitrary polynomial degree (Theorem 3.1 and Corollary 3.2), while our construction of $\mathcal{M}$ (to be found in the beginning of Section 6) gives some deeper insight into the structure of the spaces. The proof of this result is complex.

The paper is organized as follows. In Section 2 we give some preliminaries on trivariate splines, their piecewise Bernstein–Bézier representation, minimal determining sets, and smoothness conditions. In Section 3, we define uniform tetrahedral partitions Δ and state our main results. We give explicit formulae for the dimension of C^1-spline spaces of arbitrary degree on Δ. In Section 4, we introduce some notation and we rewrite the C^1-smoothness conditions of the spaces in a convenient form which is needed for the arguments developed in the subsequent sections. Section 5 contains minimal determining sets for C^1-splines on a special tetrahedral cell which consists of 24 tetrahedra. These results are used in Section 6 where we construct a suitable minimal determining set for the spline spaces and prove our main results. The paper concludes with some remarks in the final section.

2. Trivariate splines, Bernstein–Bézier representation and MDS

We briefly recall some notation well-known in multivariate spline theory (cf. [4,7,9,12]). For any integer q, we call

$$\mathcal{P}_q = \operatorname{span}\{x^i y^j z^k : i, j, k \geqslant 0,\ i + j + k \leqslant q\}$$

the $\binom{q+3}{3}$ dimensional space of trivariate polynomials of total degree q. Given a (non-degenerate) tetrahedron $T = [v_0, v_1, v_2, v_3]$ in $\mathbb{R}^3$ with vertices v_0, v_1, v_2, and v_3, the linear polynomials $\lambda_\nu \in \mathcal{P}_1,\ \nu = 0, \ldots, 3$, with the interpolation property $\lambda_\nu(v_\mu) = \delta_{\nu,\mu},\ \mu = 0, \ldots, 3$, are called the barycentric coordinates w.r.t. T. (Here, and in the following $\delta_{\nu,\mu}$ denotes Kronecker's symbol.) Every polynomial $p \in \mathcal{P}_q$ can be written in its Bernstein–Bézier representation as

$$p = \sum_{i+j+k+\ell=q} a_{i,j,k,\ell}\ B^{q,T}_{i,j,k,\ell}, \tag{2.1}$$

where

$$B^{q,T}_{i,j,k,\ell} = \frac{q!}{i!j!k!\ell!}\, \lambda_0^i \lambda_1^j \lambda_2^k \lambda_3^\ell \in \mathcal{P}_q, \qquad i+j+k+\ell = q,$$

are the Bernstein polynomials of degree q w.r.t. T. Each Bernstein–Bézier coefficient $a_{i,j,k,\ell} \in \mathbb{R}$ of p is associated with the domain point $\xi^T_{i,j,k,\ell} = (iv_0 + jv_1 + kv_2 + \ell v_3)/q$, and the set of domain points in T is denoted by $\mathcal{D}_{q,T} = \{\xi^T_{i,j,k,\ell} : i+j+k+\ell = q\}$. A point $\xi^T_{i,j,k,\ell} \in \mathcal{D}_{q,T}$ is said to be in distance m of the triangular face $[v_0, v_1, v_2]$ of T, if $\ell = m$.

We call a set of tetrahedra Δ a tetrahedral partition of a finite polyhedral domain $\Omega \subseteq \mathbb{R}^3$ if the intersection of any two different tetrahedra from Δ is a common vertex, common edge or common triangle, and the union of all tetrahedra from Δ is equal to Ω. Given a tetrahedral partition Δ of Ω and $r \in \{-1, \ldots, q-1\}$, we set

$$\mathcal{S}^r_q(\Delta) = \{s \in C^r(\Omega) : s|_T \in \mathcal{P}_q \text{ for all tetrahedra } T \in \Delta\}$$

for the space of trivariate C^r-splines of degree q w.r.t. Δ.

The coefficients $a_{\xi^T_{i,j,k,\ell}}(s) = a_{i,j,k,\ell}(s) := a_{i,j,k,\ell}(s|_T),\ i+j+k+\ell = q$, of $s \in \mathcal{S}^0_q(\Delta)$ in representation (2.1) of its polynomial pieces $s|_T \in \mathcal{P}_q,\ T \in \Delta$, are uniquely associated with the domain points in Ω which we denote by

$$\mathcal{D}_{q,\Delta} = \bigcup_{T \in \Delta} \mathcal{D}_{q,T}.$$

Given a vertex v of Δ and $T_1 = [v, v_1^1, v_2^1, v_3^1], \ldots, T_{n_v} = [v, v_1^{n_v}, v_2^{n_v}, v_3^{n_v}]$ the tetrahedra in Δ with common vertex v, for $m \in \{0, \ldots, q\}$ we call

$$\mathcal{R}^m(v) = \bigcup_{\ell=1}^{n_v} \{\xi^{T_\ell}_{q-m,i,j,k} : i+j+k = m\}$$

the ring with distance m around v. Moreover, for $m \in \{0, \ldots, q\}$, the set

$$\mathcal{D}^m(v) = \bigcup_{\ell=0}^{m} \mathcal{R}^\ell(v)$$

is called the disk of radius m around v. (As in Schumaker and Sorokina [28], we use the same terms as for bivariate splines, here. In order to avoid confusions, we note that in the trivariate setting $\mathcal{R}^m(v)$ and $\mathcal{D}^m(v)$ are sometimes called shell with distance m around v and ball of radius m around v, respectively.)

Following Alfeld et al. [2], we call $\mathcal{M} \subseteq \mathcal{D}_{q,\Delta}$ a determining set (DS) for a linear subspace $\mathcal{S}$ of $\mathcal{S}^0_q(\Delta)$, if setting the coefficients $a_\xi(s),\ \xi \in \mathcal{M}$ of a spline $s \in \mathcal{S}$ to zero, implies that $s \equiv 0$. A determining set $\mathcal{M}$ is called minimal determining set (MDS) for $\mathcal{S}$, if no determining set for $\mathcal{S}$ with fewer elements than $\mathcal{M}$ exists. Equivalently, $\mathcal{M}$ is a MDS, if the following property holds: if we set the coefficients $a_\xi(s),\ \xi \in \mathcal{M}$, of a spline $s \in \mathcal{S}$ to arbitrary values, then all its coefficients $a_\xi(s),\ \xi \in \mathcal{D}_{q,\Delta}$ are uniquely determined, i.e. s is uniquely determined. If $\mathcal{M}$ is a minimal determining set for $\mathcal{S}$, then it is obvious

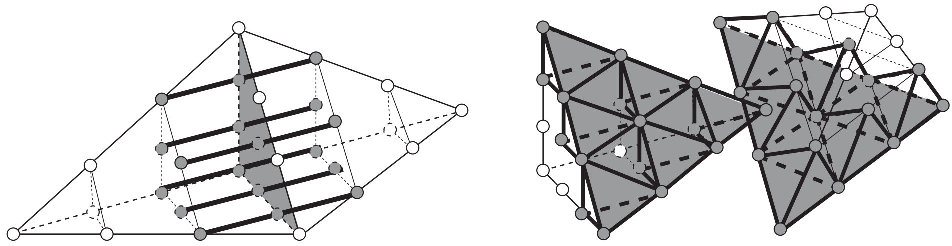

Fig. 1. Illustration of the six smoothness conditions given by Eq. (4.4) (left) and (4.7) (right) for the case of piecewise cubics, i.e. $q=3$. Smoothness conditions across the common triangular face of two neighboring tetrahedra which degenerate to univariate smoothness conditions (i.e. three coefficients are involved in each condition) are shown on the left, while the non-degenerate case (i.e. five coefficients are involved in each condition, no barycentric coordinate vanishes at the opposite vertex) is shown on the right. In both cases, the BB-coefficients associated with domain points shown as white dots are not involved in any smoothness conditions across the shaded triangular face, while the remaining BB-coefficients (illustrated as grey dots) are involved in such conditions

that $\#(\mathcal{M})$ coincides with the dimension $\dim \mathcal{S}$ of $\mathcal{S}$. (Here, and throughout the paper we denote by # the cardinality of a finite set, and by dim the dimension of a linear space.)

Given an arbitrary tetrahedral partition Δ, the dimension of $\mathcal{S}_q^0(\Delta),\ q \geqslant 1$, is easy to determine (cf. [3, Theorem 10]). In this case, it is obvious that $\mathcal{D}_{q,\Delta}$ is a MDS for $\mathcal{S}_q^0(\Delta)$ and a straight forward computation shows that

$$\dim \mathcal{S}_q^0(\Delta) = \binom{q-1}{3}\, T_\Delta + \binom{q-1}{2}\, F_\Delta + (q-1)\, E_\Delta + V_\Delta, \qquad q \geqslant 1, \tag{2.2}$$

where T_Δ is the number of tetrahedra of Δ, F_Δ is the number of triangular faces of Δ, E_Δ is the number of edges of Δ, and V_Δ is the number of vertices of Δ. (Here, and in the following we set $\binom{i}{j} := 0$, if $i < j$.) For later use, we note that if we set, in addition, V_{I} for the number of interior vertices of $\triangle$, V_{B} for the number of boundary vertices of $\triangle$, F_I for number of interior triangular faces of $\triangle$, and E_{I} for the number of interior edges of $\triangle$, then the Euler type formulae

$$\begin{aligned} V_{\mathrm{B}} &= 2\, T_\Delta - F_{\mathrm{I}} + 2, \\ T_\Delta &= V_{\mathrm{I}} - E_{\mathrm{I}} + F_{\mathrm{I}} + 1, \end{aligned} \tag{2.3}$$

hold true (see [4], for instance). The problem of determining the dimension of trivariate splines becomes more difficult if we consider subspaces $\mathcal{S}$ of $\mathcal{S}_q^0(\Delta)$ possessing smoothness conditions.

In the following, we are interested in C^1-splines, i.e. we consider the subspaces $\mathcal{S} = \mathcal{S}_q^1(\Delta),\ q \geqslant 2$ (where Δ is the tetrahedral partition of uniform type described in the next section). In order to construct minimal determining sets for these spaces, we use the well-known smoothness conditions connected with the piecewise Bernstein–Bézier representation of the splines (cf. [7,9,12]). Let $T = [v_0, v_1, v_2, v_3],\ \tilde{T} = [v_0, v_1, v_2, \tilde{v}_3] \in \Delta$, be two different tetrahedra of Δ, and suppose that $s \in \mathcal{S}_q^0(\Delta)$ is given in its piecewise representation (2.1) with coefficients $a_{i,j,k,\ell} = a_{i,j,k,\ell}(s) = a_{i,j,k,\ell}(s|_T)$ and $\tilde{a}_{i,j,k,\ell} = \tilde{a}_{i,j,k,\ell}(s) = a_{i,j,k,\ell}(s|_{\tilde{T}})$, i.e. $a_{i,j,k,0} = \tilde{a}_{i,j,k,0},\ i + j + k = q$. Then s is C^1-smooth across the common triangular face $T \cap \tilde{T} = [v_0, v_1, v_2]$ of T and $\tilde{T}$, if and only if for all $i + j + k = q - 1$,

$$\tilde{a}_{i,j,k,1} = a_{i+1,j,k,0}\ \lambda_0(\tilde{v}_3) + a_{i,j+1,k,0}\ \lambda_1(\tilde{v}_3) + a_{i,j,k+1,0}\ \lambda_2(\tilde{v}_3) + a_{i,j,k,1}\ \lambda_3(\tilde{v}_3), \tag{2.4}$$

where λ_ν, $\nu = 0, \ldots, 3$, are the barycentric coordinates w.r.t. T. Examples for these linear constraints are illustrated in Fig. 1 where the common triangular face is shaded grey, the domain points associated with the BB-coefficients involved in the smoothness conditions are shown as grey dots, and the conditions are illustrated as thick lines and small tetrahedra with thick boundary lines, respectively. It is known that the trivariate conditions (2.4) become lower-dimensional conditions if some of the involved barycentric coordinates λ_ν vanish at $\tilde{v}_3$. These are called the degenerate cases . Fig. 1 (left) shows such an example, where two barycentric coordinates are zero at $\tilde{v}_3$ and hence the smoothness conditions degenerate to conditions as in the univariate case (see Eq. (4.4) in Section 4, for instance). In the non-degenerate case (no barycentric coordinate λ_ν vanishes at $\tilde{v}_3$) each of the smoothness conditions involves 5 BB-coefficients which is shown on the right of Fig. 1 (see Eq. (4.7) in Section 4, for instance). In the next section we consider tetrahedral partitions such that for the corresponding C^1-splines only these two types of smoothness conditions appear. This is described in more detail in Section 4.

By using the piecewise Bernstein–Bézier representation of the splines the C^1-smoothness of its polynomial pieces across the common triangular face of two neighboring tetrahedra is easily described by conditions (2.4). However, if we consider a complete tetrahedral partition Δ, then the analysis of these connections becomes a complex task even in the case when Δ is of uniform type because for an overall C^1-smooth spline these are many conditions (see Section 4) which have to be simultaneously satisfied across all the four (interior) triangular faces of every tetrahedron and they cannot (in general) be considered independently.

3. Main results

In the remainder of this paper we consider a tetrahedral partition Δ of the unit cube $\Omega = [0, 1] \times [0, 1] \times [0, 1] \subseteq \mathbb{R}^3$ which is obtained as follows. Using $n + 1$ parallel planes in each of the three space dimensions we first subdivide Ω into n^3 subcubes,

$$Q_{(i,j,k)} = \left[\tfrac{i-1}{n}, \tfrac{i}{n}\right] \times \left[\tfrac{j-1}{n}, \tfrac{j}{n}\right] \times \left[\tfrac{k-1}{n}, \tfrac{k}{n}\right], \qquad i, j, k = 1, \ldots, n.$$

We let $\mathcal{F}^{[\ell]}_{(i,j,k)}$, $\ell = 1, \ldots, 6$, be the six square faces of $Q_{(i,j,k)}$, where we use the following ordering: left ($\ell = 1$), front ($\ell = 2$), bottom ($\ell = 3$), right ($\ell = 4$), back ($\ell = 5$), top ($\ell = 6$). For $i, j, k \in \{1, \ldots, n\}$ each subcube $Q_{(i,j,k)}$ is split into six square pyramids $\mathcal{P}^{[\ell]}_{(i,j,k)}$ by connecting its midpoint

$$v_{(i,j,k)} = \left(\tfrac{2i-1}{2n}, \tfrac{2j-1}{2n}, \tfrac{2k-1}{2n}\right)$$

with the vertices of the face $\mathcal{F}^{[\ell]}_{(i,j,k)}$, $\ell = 1, \ldots, 6$. Then, we insert both diagonals in each of the faces $\mathcal{F}^{[\ell]}_{(i,j,k)}$, denote their intersection point by $w^{[\ell]}_{(i,j,k)}$, and connect $v_{(i,j,k)}$ with

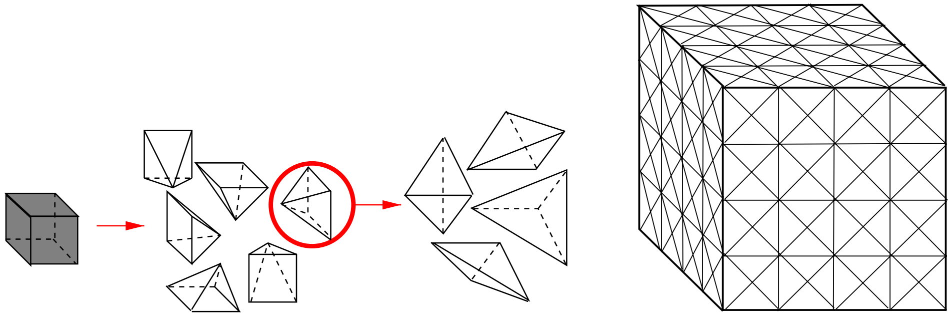

Fig. 2. The uniform type-6 tetrahedral partition Δ is obtained by subdividing each subcube $Q_{(i,j,k)}$ into 24 tetrahedra: first $Q_{(i,j,k)}$ is split into six square pyramids, then each pyramid is split into four tetrahedra (left). The intersections of Δ with certain planes parallel to the three unit planes are four-directional meshes (right)

$w^{[\ell]}_{(i,j,k)}$, $\ell = 1,\ldots,6$. This further subdivides each pyramid $\mathcal{P}^{[\ell]}_{(i,j,k)}$ into four tetrahedra, and we obtain a tetrahedral partition $\Delta_{(i,j,k)}$ of each subcube $Q_{(i,j,k)}$ which consists of 24 tetrahedra. The construction is illustrated on the right of Fig. 2. Finally, we define a tetrahedral partition Δ of Ω as

$$\Delta = \bigcup_{i,j,k\,\in\,\{1,\ldots,n\}} \Delta_{(i,j,k)}.$$

We call Δ a type-6 tetrahedral partition because for each subcube $Q_{(i,j,k)}$ the subdivision into the 24 tetrahedra described above is also obtained by slicing $Q_{(i,j,k)}$ with the six planes which contain opposite edges of $Q_{(i,j,k)}$. Alternatively, perhaps one could call Δ a nine directional (three dimensional) mesh, because (essentially) three additional planes are needed for fixing the cubes. In Carr et al. [8] the above construction is called a face-centered 24-fold subdivision of the cubes. The intersection of Δ with any plane P parallel to one of the three unit planes (in distance $\frac{\kappa}{n}$ to the origin) gives the four-directional mesh (sometimes called a uniform Δ^2 triangulation) of the intersecting square domain $P \cap \Omega$ (see Fig. 2, right) and therefore the type-6 tetrahedral partition Δ is a natural generalization of the four-directional mesh to the trivariate setting.

It is easy to see that for this uniform tetrahedral partition Δ, we have

$$\begin{aligned} T_\Delta &= 24\,n^3,\\ F_\Delta &= 48\,n^3 + 12\,n^2,\\ E_\Delta &= 29\,n^3 + 18\,n^2 + 3\,n,\\ V_\Delta &= 5\,n^3 + 6\,n^2 + 3\,n + 1 \end{aligned} \tag{3.1}$$

for the number of tetrahedra T_Δ, the number of triangular faces F_Δ, the number of edges E_Δ, and the number of vertices V_Δ of Δ, respectively. Hence, (2.2) and some elementary computations imply that

$$\dim\ \mathcal{S}^0_q(\Delta) = (4q^2+1)\,qn^3 + 6\,q^2n^2 + 3\,qn + 1, \qquad q \geqslant 1. \tag{3.2}$$

Table 1
Comparison of dimensions of splines on type-6 tetrahedral partitions for low degrees

q	$\dim \mathcal{S}_q^1(\triangle)$	$\dim \mathcal{S}_q^0(\Delta)$	$\dim \mathcal{S}_q^{-1}(\Delta)$
1	4	$5\,n^3+6\,n^2+3\,n+1$	$96\,n^3$
2	$3\,n^2+9\,n+4$	$34\,n^3+24\,n^2+6\,n+1$	$240\,n^3$
3	$6\,n^3+24\,n^2+18\,n+4$	$111\,n^3+54\,n^2+9\,n+1$	$480\,n^3$
4	$39\,n^3+66\,n^2+27\,n+4$	$260\,n^3+96\,n^2+12\,n+1$	$840\,n^3$
5	$120\,n^3+132\,n^2+36\,n+4$	$505\,n^3+150\,n^2+15\,n+1$	$1344\,n^3$
6	$273\,n^3+222\,n^2+45\,n+4$	$870\,n^3+216\,n^2+18\,n+1$	$2016\,n^3$
7	$522\,n^3+336\,n^2+54\,n+4$	$1379\,n^3+294\,n^2+21\,n+1$	$2880\,n^3$
8	$891\,n^3+474\,n^2+63\,n+4$	$2056\,n^3+384\,n^2+24\,n+1$	$3960\,n^3$
9	$1404\,n^3+636\,n^2+72\,n+4$	$2925\,n^3+486\,n^2+27\,n+1$	$5280\,n^3$

More complex arguments are needed to determine the degrees of freedom of C^1-smooth splines. In Section 6, we prove the following main result on the dimension of $\mathcal{S}_q^1(\triangle),\ q \geqslant 2$, where Δ is a type-6 tetrahedral partition.

Theorem 3.1. *The dimension of* $\mathcal{S}_q^1(\Delta)$ *is given by*

$$3\,n^2+9\,n+4, \qquad \text{if } q=2 \tag{3.3}$$

and

$$(4q^3-24q^2+53q-45)\,n^3+6\,(2q^2-7q+7)\,n^2+9\,(q-1)\,n+4, \quad \text{if } q \geqslant 3. \tag{3.4}$$

By using the result of Theorem 3.1, we explicitly compute the dimensions of the spline spaces $\mathcal{S}_q^1(\triangle),\ q \in \{2,\ldots,9\}$, i.e. for low degrees, and compare these numbers with the dimensions of the continuous and non-continuous spline spaces w.r.t. Δ (see Table 1). We observe (relatively) big differences for very small q, while it is evident that these numbers become asymptotically the same when q increases.

In the following, we give some alternative formulae for the dimension of the C^1-spline spaces w.r.t. Δ where we use the terminologies from the previous section. To do this, we note that for a type-6 tetrahedral partition Δ the number of interior vertices V_{I} and the number of boundary vertices V_{B}, respectively, are given as follows

$$V_{\mathrm{I}} = 5\,n^3-6\,n^2+3\,n-1,$$
$$V_{\mathrm{B}} = 12\,n^2+2.$$

The next corollary is obtained immediately from Theorem 3.1 and some elementary computations by using (3.1) and the Euler type formulae (2.3).

Corollary 3.2. *The dimension of* $\mathcal{S}_q^1(\Delta)$ *is given by*

$$\frac{1}{8}\left(7\,T_\triangle-2\,F_\triangle-8\,E_\triangle+32\,V_\triangle\right)=\frac{1}{8}\left(24\,V_{\mathrm{I}}+14\,V_{\mathrm{B}}-5\,T_\triangle+28\right)$$

$$= \tfrac{1}{8}\Big(9\ F_{\mathrm{I}} + 47\ V_{\mathrm{I}} - 23\ E_{\mathrm{I}} + 79\Big),$$
$$\text{if } q = 2 \tag{3.5}$$

and

$$\begin{aligned}
&\tfrac{1}{12}\ \Big((2q^3 - 36q^2 + 175q - 219)\ T_\triangle + 12\ (q^2 - 8q + 12)\ F_\triangle \\
&\quad + 12\ (3q - 7)\ E_\triangle + 48\ V_\triangle\Big) \\
&\quad = \tfrac{1}{12}\Big(36\ (q-1)\ V_{\mathrm{I}} + 12\ (q^2 - 2q + 2)\ V_{\mathrm{B}} \\
&\qquad +(2q^3 - 12q^2 + 19q - 15)\ T_\triangle - 12\ (2q^2 - 7q + 3)\Big) \\
&\quad = \tfrac{1}{12}\Big((2q^3 - 5q + 9)\ F_{\mathrm{I}} + (2q^3 + 12q^2 + 7q - 3)\ V_{\mathrm{I}} \\
&\qquad -(2q^3 + 12q^2 - 29q + 33)\ E_{\mathrm{I}} + 2q^3 + 12q^2 + 7q + 45\Big), \\
&\qquad \text{if } q \geqslant 3.
\end{aligned} \tag{3.6}$$

4. Domain points and C^1-conditions on type-6 tetrahedral partitions

For proving our main result (Theorem 3.1) we have to analyze the spline spaces $\mathcal{S}_q^1(\varDelta)$, where $q \geqslant 2$. This is done by constructing an appropriate MDS $\mathcal{M}$ (see Section 6) for the splines on the partition $\varDelta$ introduced in the previous section. By the nature of the problem the choice of points in $\mathcal{M}$ is sometimes non-symmetric and hence we need a tool to conveniently access individual domain points from $\mathcal{D}_{q,\varDelta}$ within the tetrahedra of the different cubes. In this section, we develop such a tool. We introduce a terminology which allows us to describe the set $\mathcal{M}$ for splines of arbitrary degrees (including the cases of quadratic, cubic and quartic splines). In particular, we use this specific notation to rewrite the smoothness conditions (2.4) for the splines on type-6 tetrahedral partitions $\varDelta$ in a convenient form, such that the subsequent proofs can be kept of moderate length. It generalizes a description of domain points and smoothness conditions to the trivariate setting which was introduced in the scattered data fitting method of Davydov and Zeilfelder [11] for bivariate splines on the four-directional mesh.

For all $i, j, k \in \{1, \ldots, n\}$ we set for the ring with distance $m \in \{0, \ldots, q\}$ around the midpoint $v_{(i,j,k)}$ of $Q_{(i,j,k)}$,

$$\mathcal{R}^m(v_{(i,j,k)}) = \bigcup_{\rho \in \{0,2m\}} \bigcup_{\substack{\sigma,\tau \in \{0,\ldots,2m\} \\ \sigma+\tau \text{ even}}} \bigcup_{\nu=1,2,3} \Big(\Big\{\xi \in \mathcal{D}_{q,\varDelta_{(i,j,k)}} :\ \xi = \xi^{m,[\pi_\nu(\rho,\sigma,\tau)]}_{(i,j,k)} = v_{(i,j,k)} + \frac{\pi_\nu(\rho,\sigma,\tau) - (m,m,m)}{2qn}\Big\}\Big), \tag{4.1}$$

where here and in the following we use the abbreviations

$$\pi_1(a,b,c) := (a,b,c),\ \ \pi_2(a,b,c) := (b,a,c),\ \ \pi_3(a,b,c) := (b,c,a).$$

The idea is to consider the $4\,q^3+6\,q^2+4\,q+1$ domain points from $\mathcal{D}_{q,\Delta_{(i,j,k)}}$ in the cube $Q_{(i,j,k)}$ as points which are organized on the boundary of $q+1$ individual subcubes around $v_{(i,j,k)}$ in $Q_{(i,j,k)}$. The index m of $\xi = \xi^{m,[\pi_\nu(\rho,\sigma,\tau)]}_{(i,j,k)}$ indicates the distance m of ξ to the midpoint $v_{(i,j,k)}$, and is associated with the boundary of the mth subcube. Hence, the case $m = 0$ degenerates to a subcube which exists of exactly one point, i.e. the point $v_{(i,j,k)}$, while the case $m = q$ describes all the domain points which lie on the boundary of $Q_{(i,j,k)}$, which is the qth subcube. Moreover, there are $q-1$ additional subcubes around $v_{(i,j,k)}$ which are in between these two cases. The choice of ρ and π_ν, $\nu = 1, 2, 3$, determines on which square face of the boundary of the subcubes a point ξ is placed. More precisely, $\rho = 0$ and $\nu = 1$ means that the corresponding points lie on the left (boundary) square face of the subcubes (i.e. in the pyramid $\mathcal{P}^{[1]}_{(i,j,k)}$), while $\rho = 2m$ and $\nu = 1$ means that the corresponding points lie on the right (boundary) square face of the subcubes (i.e. in the pyramid $\mathcal{P}^{[4]}_{(i,j,k)}$). Similarly, the choice $\nu = 2$ describes points on the front ($\rho = 0$) and back ($\rho = 2m$) faces of the subcubes (i.e. in the pyramid $\mathcal{P}^{[2]}_{(i,j,k)}$ and $\mathcal{P}^{[5]}_{(i,j,k)}$, respectively), while $\nu = 3$ includes all the domain points on the bottom ($\rho = 0$) and top ($\rho = 2m$) faces of the subcubes (i.e. in the pyramid $\mathcal{P}^{[3]}_{(i,j,k)}$ and $\mathcal{P}^{[6]}_{(i,j,k)}$, respectively).

We try to illustrate the introduced terminology in Fig. 3, for the case $q = 3$. In order to draw the domain points on the boundary of each of the four subcubes simultaneously, we first map the six boundary faces of each subcube into the plane as illustrated in the top of Fig. 3 (here, $[\ell]$ indicates the pyramids $\mathcal{P}^{[\ell]}_{(i,j,k)}$which intersect the corresponding square face, the cube is unfolded so that the left square face, i.e. $[\ell] = [1]$, is in the middle of the cross, for instance). Then, we show the domain points on the boundary of the four subcubes for $m = 0, 1, 2, 3$, where we draw the points by indicating $\xi = \xi^{m,[\pi_\nu(\rho,\sigma,\tau)]}_{(i,j,k)}$ with $[\pi_\nu(\rho, \sigma, \tau)]$. In addition, we add the diagonals obtained from the subdivision of the cubes into the 24 tetrahedra. By the nature of the above mapping some of the domain points appear more than once, and therefore we show the essential points using grey boxes.

For later use, we call the domain points inside the square face $\mathcal{F}^{[\nu]}_{(i,j,k)}$ of $Q_{(i,j,k)}$, i.e. the points ξ of the form $\xi = \xi^{q,[\pi_\nu(0,\sigma,\tau)]}_{(i,j,k)}$, $\sigma, \tau \in \{0, \ldots, 2q\}$, $\sigma+\tau$ even, points at a distance zero of $\mathcal{F}^{[\nu]}_{(i,j,k)}$, where $\nu = 1, 2, 3$. In addition, we call the domain points which are on the next layer away from the square face $\mathcal{F}^{[\nu]}_{(i,j,k)}$ of $Q_{(i,j,k)}$, i.e. the points ξ of the form $\xi = \xi^{q-1,[\pi_\nu(0,\sigma,\tau)]}_{(i,j,k)}$, $\sigma, \tau \in \{0, \ldots, 2(q-1)\}$, $\sigma+\tau$ even, or $\xi = \xi^{q,[\pi_\nu(1,\sigma+1,\tau)]}_{(i,j,k)}$ or $\xi = \xi^{q,[\pi_\nu(1,\tau,\sigma+1)]}_{(i,j,k)}$, $\sigma \in \{0, \ldots, 2(q-1)\}$, σ even, $\tau \in \{0, 2q\}$, or $\xi = \xi^{q,[\pi_\nu(2,\sigma,\tau)]}_{(i,j,k)}$, $\sigma \in \{0, 2q\}$, $\tau \in \{\sigma, 2q-\sigma\}$, points at a distance one of $\mathcal{F}^{[\nu]}_{(i,j,k)}$, where $\nu = 1, 2, 3$. In Fig. 4, again we consider the case $q = 3$, use the above mapping, and show the domain points in distance one and zero of the square faces $\mathcal{F}^{[\ell]}_{(i,j,k)}$, $\ell = 1, 2, 3$. Obviously, in this case these points are on the rings $\mathcal{R}^m(v_{(i,j,k)})$, where $m \in \{2, 3\}$. In this figure the domain points are shown as dots (containing various symbols) and we indicate the essential points by using grey boxes. The points at a distance zero and one of $\mathcal{F}^{[1]}_{(i,j,k)}$ are shown as grey dots, while the points at a distance zero and one of $\mathcal{F}^{[2]}_{(i,j,k)}$ and $\mathcal{F}^{[3]}_{(i,j,k)}$ are indicated by large circles and crosses, respectively.

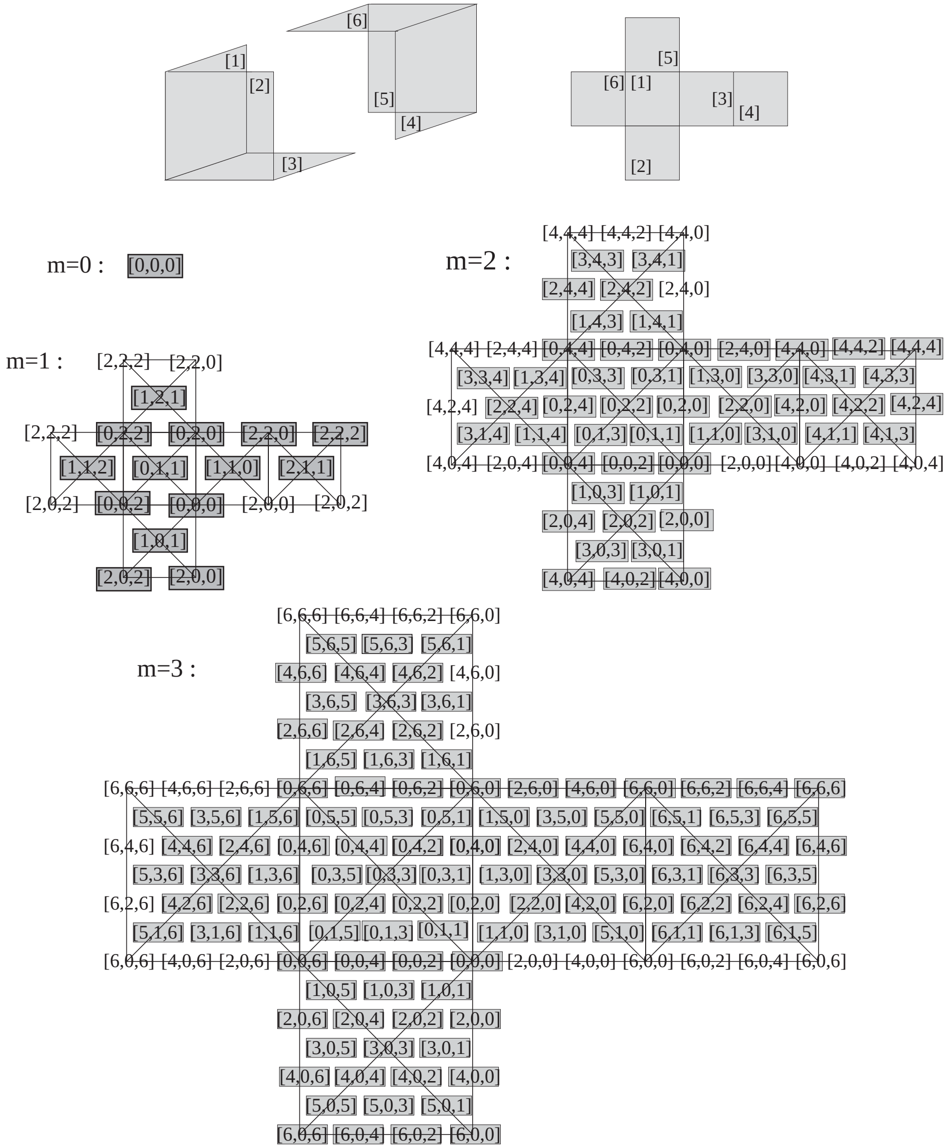

Fig. 3. Indexing the domain points within a cube $Q_{(i,j,k)}$ for $q = 3$. The boundary faces of the four subcubes, i.e. the rings $\mathcal{R}^m(v_{(i,j,k)})$, contain 1, 14, 50, and 110 domain points for $m = 0, 1, 2, 3$, respectively. These points are indicated by showing their index $[\pi_v(\rho, \sigma, \tau)]$

In the following, we set $a_\xi = a^{m,[\pi_v(\rho,\sigma,\tau)]}_{(i,j,k)}$, if $\xi = \xi^{m,[\pi_v(\rho,\sigma,\tau)]}_{(i,j,k)}$, for the coefficients $a_\xi = a_\xi(s),\ \xi \in \mathcal{D}_{q,\Delta}$, of a spline $s \in \mathcal{S}^1_q(\triangle),\ q \geqslant 2$, on a type-6 tetrahedral partition Δ, and we proceed by rewriting the (remaining) continuity and smoothness conditions (2.4) for the spline spaces. For $i, j, k \in \{1, \ldots, n\}$, the continuity of s on the common triangles

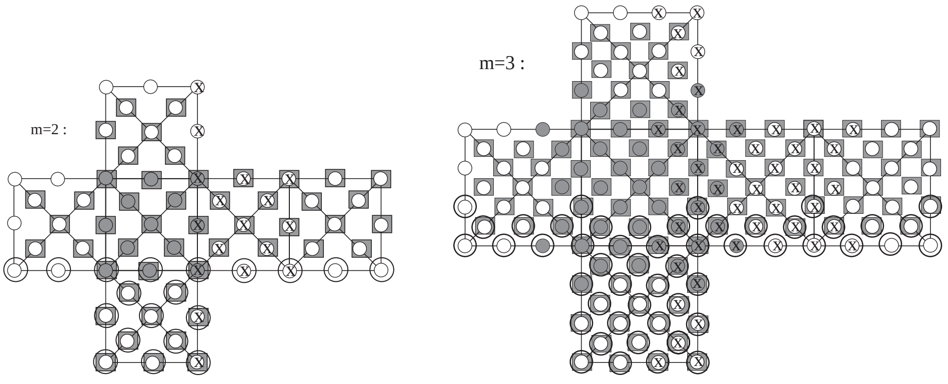

Fig. 4. The sets of domain points at a distance one (left) and zero (right) of the left, front and bottom faces of a cube in the case $q = 3$

of the faces $\mathcal{F}^{[1]}_{(i,j,k)}$ and $\mathcal{F}^{[4]}_{(i-1,j,k)}$, $i \neq 1$, $\mathcal{F}^{[2]}_{(i,j,k)}$ and $\mathcal{F}^{[5]}_{(i,j-1,k)}$, $j \neq 1$, and $\mathcal{F}^{[3]}_{(i,j,k)}$ and $\mathcal{F}^{[6]}_{(i,j,k-1)}$, $k \neq 1$, respectively, implies that

$$a_\alpha = a_\beta, \tag{4.2}$$

where

$$\alpha = \xi^{q,[\pi_v(2q,\sigma,\tau)]}_{(i,j,k)-e_v} \text{ and } \beta = \xi^{q,[\pi_v(0,\sigma,\tau)]}_{(i,j,k)}, \tag{4.3}$$

$\sigma, \tau \in \{0, \ldots, 2q\}$, $\sigma + \tau$ even, $v \in \{1, 2, 3\}$, and $e_v = (\delta_{v,\mu})_{\mu=1,2,3}$. For $i, j, k \in \{1, \ldots, n\}$, the C^1-smoothness conditions of s across the common triangular faces of $\mathcal{F}^{[1]}_{(i,j,k)}$ and $\mathcal{F}^{[4]}_{(i-1,j,k)}$, $i \neq 1$, $\mathcal{F}^{[2]}_{(i,j,k)}$ and $\mathcal{F}^{[5]}_{(i,j-1,k)}$, $j \neq 1$, and $\mathcal{F}^{[3]}_{(i,j,k)}$ and $\mathcal{F}^{[6]}_{(i,j,k-1)}$, $k \neq 1$, respectively, are given as

$$a_\alpha = \frac{1}{2}\,(a_\beta + a_\gamma), \tag{4.4}$$

where

$$\alpha = \xi^{q,[\pi_v(0,\sigma,\tau)]}_{(i,j,k)},\ \beta = \xi^{q-1,[\pi_v(0,\sigma-1,\tau-1)]}_{(i,j,k)},\ \text{and } \gamma = \xi^{q-1,[\pi_v(2q-2,\sigma-1,\tau-1)]}_{(i,j,k)-e_v}, \tag{4.5}$$

$\sigma, \tau \in \{1, \ldots, 2q-1\}$, $\sigma + \tau$ even, and $v \in \{1, 2, 3\}$. We observe that the C^1-smoothness conditions of s across the common triangular faces of the four tetrahedra inside each pyramid $\mathcal{P}^{[\ell]}_{(i,j,k)}$, $\ell = 1, \ldots, 6$, are given by the same Eq. (4.4). More precisely, for $i, j, k \in \{1, \ldots, n\}$, (4.4) holds for all α, β, and γ of the form

$$\alpha = \xi^{m,[\pi_v(\rho,\sigma,\sigma)]}_{(i,j,k)},\ \beta = \xi^{m,[\pi_v(\rho,\sigma-1,\sigma+1)]}_{(i,j,k)},\ \text{and } \gamma = \xi^{m,[\pi_v(\rho,\sigma+1,\sigma-1)]}_{(i,j,k)}$$

and

$$\alpha = \xi^{m,[\pi_v(\rho,\sigma,2m-\sigma)]}_{(i,j,k)},\ \beta = \xi^{m,[\pi_v(\rho,\sigma-1,2m-\sigma-1)]}_{(i,j,k)},\ \text{and}$$
$$\gamma = \xi^{m,[\pi_v(\rho,\sigma+1,2m-\sigma+1)]}_{(i,j,k)}, \tag{4.6}$$

respectively, where $\rho \in \{0, 2m\}$, $\sigma \in \{1, \ldots, 2m-1\}$, $m = 1, \ldots, q$, and $v \in \{1, 2, 3\}$. The C^1-smoothness conditions (4.5) and (4.6) of s are illustrated in Fig. 1 (left) by showing the case of cubic splines (i.e. $q = 3$), where we symbolize the domain points associated with the involved Bernstein–Bézier coefficients by grey dots, while others are shown as white dots. For fixed $m \in \{1, \ldots, q\}$ the conditions described by (4.4), where α, β, γ are as in (4.6), involve Bernstein–Bézier coefficients where the associated domain points are on the ring $\mathcal{R}^m(v_{(i,j,k)})$, only. Obviously, these conditions are of univariate type. In addition, for the partition Δ there are C^1-smoothness conditions involving Bernstein–Bézier coefficients associated with domain points on the rings $\mathcal{R}^m(v_{(i,j,k)})$ and $\mathcal{R}^{m-1}(v_{(i,j,k)})$, simultaneously. These are the C^1-smoothness conditions of s across the common triangular faces of tetrahedra contained in different pyramids $\mathcal{P}^{[\ell]}_{(i,j,k)}$, $\mathcal{P}^{[\ell']}_{(i,j,k)}$, $\ell \neq \ell'$, of the same cube $Q_{i,j,k}$. These (non-degenerate) conditions involve five coefficients and are illustrated in Fig. 1 (right)—again this figure deals with the case of cubics, the involved Bernstein–Bézier coefficients are shown as grey dots, and the others are shown as white dots.

For $i, j, k \in \{1, \ldots, n\}$, the C^1-smoothness conditions of s across the common triangular faces of different pyramids in $Q_{i,j,k}$ are of the form

$$a_\alpha = (a_\beta + a_\gamma) - \tfrac{1}{2}(a_\zeta + a_\eta), \tag{4.7}$$

where $\alpha, \beta, \gamma, \zeta$ and η are given as

$$\begin{aligned}
&\alpha = \xi^{m-1,[\pi_v(\sigma,0,0)]}_{(i,j,k)},\ \beta = \xi^{m,[\pi_v(\sigma+1,1,0)]}_{(i,j,k)},\ \gamma = \xi^{m,[\pi_v(\sigma+1,0,1)]}_{(i,j,k)},\\
&\zeta = \xi^{m,[\pi_v(\sigma,0,0)]}_{(i,j,k)},\ \text{and } \eta = \xi^{m,[\pi_v(\sigma+2,0,0)]}_{(i,j,k)},\\
&\alpha = \xi^{m-1,[\pi_v(\sigma,2(m-1),0)]}_{(i,j,k)},\ \beta = \xi^{m,[\pi_v(\sigma+1,2m,1)]}_{(i,j,k)},\ \gamma = \xi^{m,[\pi_v(\sigma+1,2m-1,0)]}_{(i,j,k)},\\
&\zeta = \xi^{m,[\pi_v(\sigma,2m,0)]}_{(i,j,k)},\ \text{and } \eta = \xi^{m,[\pi_v(\sigma+2,2m,0)]}_{(i,j,k)},\\
&\alpha = \xi^{m-1,[\pi_v(\sigma,0,2(m-1))]}_{(i,j,k)},\ \beta = \xi^{m,[\pi_v(\sigma+1,1,2m)]}_{(i,j,k)},\ \gamma = \xi^{m,[\pi_v(\sigma+1,0,2m-1)]}_{(i,j,k)},\\
&\zeta = \xi^{m,[\pi_v(\sigma,0,2m)]}_{(i,j,k)},\ \text{and } \eta = \xi^{m,[\pi_v(\sigma+2,0,2m)]}_{(i,j,k)}
\end{aligned}$$

and

$$\begin{aligned}
&\alpha = \xi^{m-1,[\pi_v(\sigma,2(m-1),2(m-1)]}_{(i,j,k)},\ \beta = \xi^{m,[\pi_v(\sigma+1,2m,2m-1)]}_{(i,j,k)},\\
&\gamma = \xi^{m,[\pi_v(\sigma+1,2m-1,2m)]}_{(i,j,k)},\\
&\zeta = \xi^{m,[\pi_v(\sigma,2m,2m)]}_{(i,j,k)},\ \text{and } \eta = \xi^{m,[\pi_v(\sigma+2,2m,2m)]}_{(i,j,k)},
\end{aligned} \tag{4.8}$$

respectively, where $\sigma \in \{0, \ldots, 2(m-1)\}$, σ even, $m = 1, \ldots, q$, and $v \in \{1, 2, 3\}$.

5. Minimal determining sets for C^1-splines on a tetrahedral cell

We consider the spaces $\mathcal{S}^1_q(\Delta_{(1,1,1)})$, $q \geqslant 2$, where $\Delta_{(1,1,1)}$ is obtained from subdividing the cube $Q := Q_{(1,1,1)}$ into 24 tetrahedra. This is the case $n = 1$ in Section 3 and $\Delta_{(1,1,1)}$ is a tetrahedral cell with one interior vertex $v := v_{(1,1,1)}$. This can be considered as the starting point of our inductive method for proving our main result (Theorem 3.1) which is presented in Section 6. In the following, we give two different MDS for the spaces $\mathcal{S}^1_q(\Delta_{(1,1,1)})$, $q \geqslant 2$, which we denote by $\widetilde{\mathcal{M}}_Q$ and $\mathcal{M}_Q$, respectively. The choice of points in the first set $\widetilde{\mathcal{M}}_Q$ is quite symmetric, and the basic idea here is, roughly speaking, that we choose points $\xi^{m,[\pi_\nu(\rho,\sigma,\tau)]} := \xi^{m,[\pi_\nu(\rho,\sigma,\tau)]}_{(1,1,1)}$ on the rings $\mathcal{R}^m(v)$ by working from the interior to the boundary of Q, i.e. we consider the rings $\mathcal{R}^m(v)$ in the order $m = 1, \ldots, q$. Computing the cardinality of $\widetilde{\mathcal{M}}_Q$, we determine the dimension of $\mathcal{S}_q^1(\Delta_{(1,1,1)}),\ q \geqslant 2$. The second MDS $\mathcal{M}_Q$ for $\mathcal{S}_q^1(\Delta_{(1,1,1)}),\ q \geqslant 2$, is more complex than $\widetilde{\mathcal{M}}_Q$ since it possesses fewer symmetries. The below proof (of Theorem 5.3) shows that for this set different arguments are necessary. In this case, our inductive proof works from the boundary of three square faces of Q to the interior, and then—using induction again—from the interior to the boundary of the three remaining faces of Q. We take advantage of the fact that at this point it is sufficient to show that $\mathcal{M}_Q$ is a DS. Moreover, on the other hand, $\mathcal{M}_Q$ is chosen such that it allows us to deal with C^1-splines, where the values as well as its first derivatives are already determined across certain square faces at the boundary of Q, and therefore we need $\mathcal{M}_Q$ for the construction of the MDS $\mathcal{M}$ for the whole spline spaces $\mathcal{S}_q^1(\triangle),\ q \geqslant 2$. In fact, the set $\mathcal{M}_Q$ is the key to building up the construction for the whole space which we present in Section 6. Note that both MDS give some insight into the structure of the trivariate spline spaces.

In the following, we define $\widetilde{\mathcal{M}}_Q \subseteq \mathcal{D}_{q,\Delta_{(1,1,1)}}$. To do this, we need some auxiliary sets which we denote by $\mathcal{D}$, $\Lambda^m(v)$, and $\Theta^m(v),\ m = 2, \ldots, q$. First, $\mathcal{D} \subseteq \mathcal{R}^1(v)$ is a simple set which determines the points from the disk with radius 1 around v. We set

$$\mathcal{D} := \{\xi^{1,[0,0,0]}\} \cup \{\xi^{1,[\pi_\nu(2,0,0)]},\ \nu = 1, 2, 3\}. \tag{5.1}$$

Hence, $\mathcal{D}$ contains the points which are shown as black dots for the case $m = 1$ in Fig. 5. In this figure, we use the same mapping for the different rings as in the top of Fig. 3. Again, we show the case $q = 3$, here, and since some of the domain points appear more than once, we indicate the essential points by using grey boxes. For $m \in \{2, \ldots, q\}$, we set

$$\Lambda^m(v) := \bigcup_{\rho \in \{0,2m\}} \bigcup_{\substack{\sigma \in \{0,\ldots,2m\} \\ \tau \in \{\sigma, 2m-\sigma\}}} \bigcup_{\nu=1,2,3} \{\xi^{m,[\pi_\nu(\rho,\sigma,\tau)]}\}.$$

These sets describe the points on the diagonals of the boundary of the subcubes (see previous section) associated with the rings $\mathcal{R}^m(v),\ m = 2, \ldots, q$. In Fig. 5, we show them as grey dots (in the cases $m = 2$ and 3). Moreover, we let $\Theta^2(v) := \emptyset$ and for $m \in \{3, \ldots, q\}$, we set

$$\Theta^m(v) := \bigcup_{\substack{\rho \in \{2,\ldots,2(m-1)-2\} \\ \rho \text{ even}}} \bigcup_{\substack{\sigma \in \{0,2m-1\} \\ \tau \in \{\sigma, 2m-\sigma\}}} \bigcup_{\nu=1,2,3} \{\xi^{m,[\pi_\nu(\rho+1,\sigma+1,\tau)]}\}.$$

These sets describe certain points being at a distance one to some of the interior triangular faces of $\Delta_{(1,1,1)}$ with two vertices of Q. To describe this differently, one can say that these points are on the boundary of certain subcubes (see previous section) associated with the rings $\mathcal{R}^m(v),\ m = 3, \ldots, q$, and close to the edges of these subcubes. In Fig. 5, we show them as white dots (in the case $m = 3$).

Roughly speaking, the set $\widetilde{\mathcal{M}}_Q$ is now essentially defined by removing the points on the four interior triangular faces of each pyramid in Q (grey dots in Fig. 5) and certain points which lie at a distance one to the remaining interior triangular faces of Q (white dots in

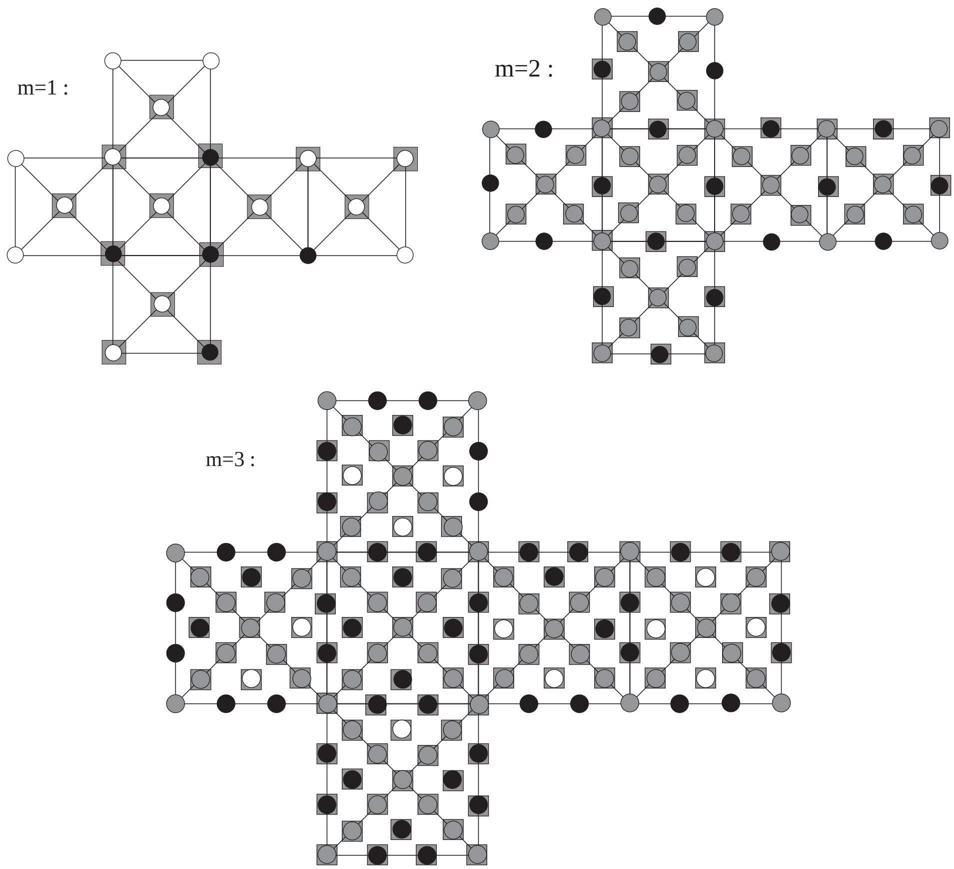

Fig. 5. The choice of points for $\widetilde{\mathcal{M}}_Q$ in the case $q = 3$. The figure shows the rings $\mathcal{R}^m(v)$, $m = 1$ (top, left), $m = 2$ (top,right), and $m = 3$ (bottom), where the points in $\widetilde{\mathcal{M}}_Q$ are marked by black dots

Fig. 5), and adding the points from $\mathcal{D}$. In the example of Fig. 5, $\widetilde{\mathcal{M}}_Q$ consists of all the points shown as black dots surrounded by grey boxes. Here, we have $q = 3$ and the cardinality of $\widetilde{\mathcal{M}}_Q$ is equal to $4 + 12 + 36 = 52$. More precisely, we define

$$\widetilde{\mathcal{M}}_Q := \mathcal{D} \cup \bigcup_{m=2}^{q} \Big(\mathcal{R}^m(v) \backslash (\Lambda^m(v) \cup \Theta^m(v))\Big). \tag{5.2}$$

Theorem 5.1. *The set $\widetilde{\mathcal{M}}_Q$ is a minimal determining set for $\mathcal{S}^1_q(\triangle_{(1,1,1)})$, $q \geqslant 2$.*

Proof. Let arbitrary coefficients $a_\xi = a_\xi(s)$, $\xi \in \widetilde{\mathcal{M}}_Q$, of a spline $s \in \mathcal{S}^1_q(\triangle_{(1,1,1)})$, $q \geqslant 2$, be given. We have to show that all coefficients of s, i.e. the coefficients a_ξ, where $\xi \in D_{q,\Delta_{(1,1,1)}} = \mathcal{D}^q(v)$, are uniquely determined, while all the C^1-smoothness conditions

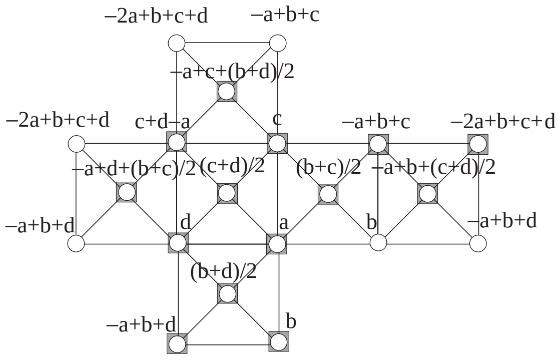

Fig. 6. Computation of coefficients associated with points in $\mathcal{R}^1(v)$

of the form (4.4), where α, β, and γ are as in (4.6), and (4.7), where α, β, γ, ζ, and η are as in (4.8), are satisfied.

First let us note, that the choice of $\mathcal{D}$ uniquely determines all the coefficients associated with points in the disk $\mathcal{D}^1(v)$. This easily follows from some elementary computations using the 24 C^1-smoothness conditions involving the coefficients associated with points from that disk, only. The results of these computations are illustrated in Fig. 6, where we set $a := a_\xi,\ \xi = \xi^{1,[0,0,0]},\ b := a_\xi,\ \xi = \xi^{1,[2,0,0]},\ c := a_\xi,\ \xi = \xi^{1,[0,2,0]}$, and $d := a_\xi,\ \xi = \xi^{1,[0,0,2]}$, and compute the remaining coefficients from $\mathcal{R}^1(v)$. In addition, we have $a_\xi = (-a + b + c + d)/2,\ \text{if } \xi = \xi^{0,[0,0,0]}$.

We now claim that the coefficients $a_\xi,\ \xi \in \mathcal{D}^m(v)$, are uniquely determined for $m \in \{2, \ldots, q\}$. To show this we use induction.

The case $m = 2$ differs somewhat from the remaining cases, and therefore we first consider this case. Here, we have to show that the coefficients $a_\xi,\ \xi \in \mathcal{R}^2(v)$, are uniquely determined. Since the points $\xi^{2,[\pi_v(2,\sigma,\tau)]},\ \sigma, \tau \in \{0, 4\},\ v = 1, 2, 3$, are contained in $\widetilde{\mathcal{M}}_Q$, it follows from a standard argument known from bivariate spline theory (cf. [27]) involving the C^1-smoothness conditions of the form (4.4), where α, β, and γ are as in (4.6) and $m = 2$, that a_ξ is uniquely determined if $\xi \in \Lambda^2(v) \setminus \{\xi^{2,[\rho,\sigma,\tau]} :\ \rho, \sigma, \tau \in \{0, 4\}\}$. As we have seen above, the coefficients of s associated with points from $\mathcal{D}^1(v)$ are uniquely determined, and hence we can now apply the C^1-smoothness conditions of the form (4.7), where α, β, γ, ζ, and η is as in (4.8) and $m = 2$, which determine the remaining coefficients on $\mathcal{R}^2(v)$, i.e. the coefficients a_ξ, where $\xi = \xi^{2,[\rho,\sigma,\tau]},\ \rho, \sigma, \tau \in \{0, 4\}$. Any of these coefficients a_ξ (which correspond to one of the eight corners of the subcube associated with $\mathcal{R}^2(v)$) is involved in three C^1-smoothness conditions of the latter form, but we observe that independent of which condition is chosen the value of a_ξ is always the same. Hence, a_ξ is uniquely determined. For instance, we compute $a_{\xi^{2,[0,0,0]}} = a_{\xi^{2,[2,0,0]}} + a_{\xi^{2,[0,2,0]}} + a_{\xi^{2,[0,0,2]}} - 2\, a_{\xi^{1,[0,0,0]}}$. We conclude that the coefficients $a_\xi,\ \xi \in \mathcal{D}^2(v)$ are uniquely determined.

Let us assume that we have already shown that the coefficients $a_\xi,\ \xi \in \mathcal{D}^{m-1}(v)$, where $m \in \{3, \ldots, q\}$ are uniquely determined. We now prove that the coefficients $a_\xi,\ \xi \in \mathcal{R}^m(v)$ are uniquely determined. To do this, let us note first that it is obvious that all the points in the

Table 2
Comparison of dimensions of splines on the tetrahedral cell $\Delta_{(1,1,1)}$

q	dim $\mathcal{S}_q^1(\Delta_{(1,1,1)})$	dim $\mathcal{S}_q^0(\Delta_{(1,1,1)})$	dim $\mathcal{S}_q^{-1}(\Delta_{(1,1,1)})$
1	4	14	96
2	16	65	240
3	52	175	480
4	136	369	840
5	292	671	1344
6	548	1105	2016
7	916	1695	2880
8	1432	2465	3960
9	2116	3439	5280

interior of the edges of the subcubes associated with the ring $\mathcal{R}^m(v)$ are contained in $\widetilde{\mathcal{M}}_Q$. Moreover, we have $\xi^{m,[\pi_v(\rho+1,\sigma,\tau)]} \in \widetilde{\mathcal{M}}_Q$, where $\rho \in \{2,\ldots,2(m-1)-2\}$, ρ even, $\sigma \in \{0, 2m-1\}$, $\tau \in \{\sigma+1, 2m-1-\sigma\}$, $v = 1,2,3$. (For the case $m = 3$, these are the black dots in the interior of the triangles shown in Fig. 5.) From the induction hypothesis, we know that the coefficients associated with the domain points in $\mathcal{R}^{m-1}(v)$ are uniquely determined, and therefore it follows from (4.7), where $\alpha, \beta, \gamma, \zeta$, and η are as in (4.8) and $\sigma \in \{2,\ldots,2(m-1)-2\}$, σ even, that the coefficients a_ξ, $\xi \in \Theta^m(v)$, are uniquely determined. Moreover, the remaining points in the interior of the triangles on the square faces of the subcubes associated with the ring $\mathcal{R}^m(v)$ are contained in $\widetilde{\mathcal{M}}_Q$, and hence the C^1-smoothness conditions (4.4), where α, β, and γ are as in (4.6) and $\rho \in \{0, 2m\}$, $\sigma \in \{1,\ldots,2m-1\}$, imply that a_ξ is uniquely determined if $\xi \in \Lambda^m(v)\backslash\{\xi^{m,[\rho,\sigma,\tau]} : \rho,\sigma,\tau \in \{0,2m\}\}$. In particular, by using the argument from the bivariate theory mentioned above, the coefficients a_ξ, where $\xi = \xi^{m,[\pi_v(\rho,m,m)]}$, $\rho \in \{0,2m\}$, $v = 1,2,3$, are uniquely determined. Any of the coefficients a_ξ, $\xi = \xi^{m,[\rho,\sigma,\tau]}$, $\rho,\sigma,\tau \in \{0,2m\}$ (which correspond to one of the eight corners of the subcube associated with $\mathcal{R}^m(v)$) is involved in three C^1-smoothness conditions which we have not used, yet, i.e. conditions of the form (4.7), where $\alpha, \beta, \gamma, \zeta$, and η are as in (4.8) and $\sigma \in \{0, 2(m-1)\}$. By the induction hypothesis, the coefficients a_ξ, where $\xi = \xi^{m-1,[\rho,\sigma,\tau]}$, $\rho,\sigma,\tau \in \{0,2(m-1)\}$ are already uniquely determined, and therefore the same argument as in the case $m = 2$ shows that the coefficients a_ξ, $\xi = \xi^{m,[\rho,\sigma,\tau]}$, $\rho,\sigma,\tau \in \{0,2m\}$ are uniquely determined. We conclude that the coefficients a_ξ, $\xi \in \mathcal{R}^m(v)$ and hence the coefficients a_ξ, $\xi \in \mathcal{D}^m(v)$ are uniquely determined.

The proof of the theorem is complete. □

The next result is obtained by counting the number of points in the minimal determining set $\widetilde{\mathcal{M}}_Q$ for $\mathcal{S}_q^1(\triangle_{(1,1,1)})$, $q \geqslant 2$, defined in (5.2). In Table 2 we compare the dimension of these spaces with the dimensions of continuous and non-continuous splines on the same tetrahedral cell $\triangle_{(1,1,1)}$.

Theorem 5.2. *The dimension of* $\mathcal{S}_q^1(\triangle_{(1,1,1)})$, $q \geqslant 2$, *is given by* $4\,(q^3 - 3q^2 + 5q - 2)$.

Proof. Let us denote by d_m the number of points in $\mathcal{D}^m(v) \cap \widetilde{\mathcal{M}}_Q$, $m = 2,\ldots,q$. A simple count shows that there are $m-1$ points from $\widetilde{\mathcal{M}}_Q$ on each of the twelve edges of the subcubes associated with $\mathcal{R}^m(v)$. Moreover, twelve triangles on the square faces of these

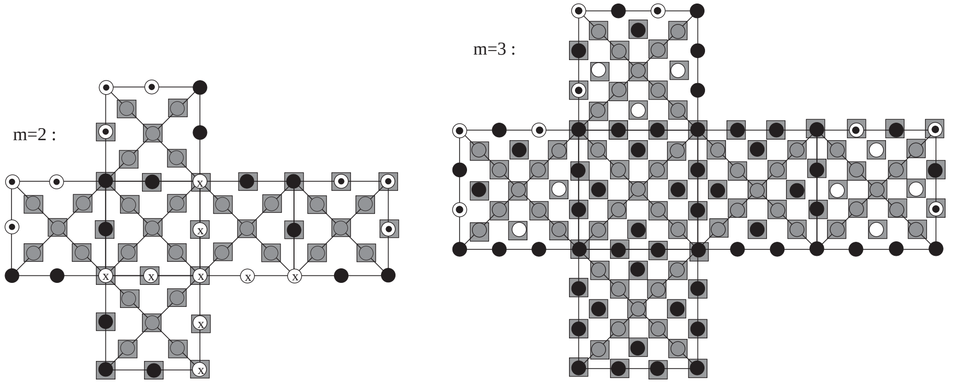

Fig. 7. The choice of points for $\mathcal{M}_Q$ in the case $q = 3$. The figure shows the rings $\mathcal{R}^2(v)$ (left) and $\mathcal{R}^3(v)$ (right), where the points of $\mathcal{M}_Q$ are marked by black dots. These are all the points of $\mathcal{M}_Q$, since $\mathcal{D}^1(v) \cap \mathcal{M}_Q = \emptyset$

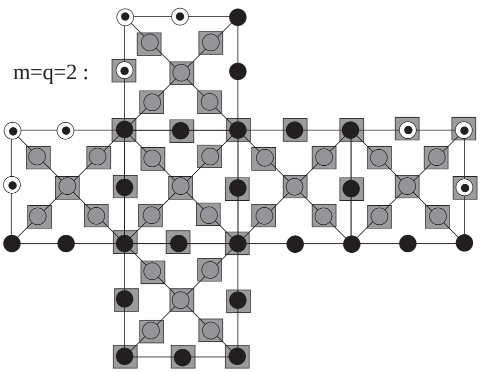

Fig. 8. The choice of points for $\mathcal{M}_Q$ in the case $q = 2$. The figure shows the ring $\mathcal{R}^2(v)$, where the points of $\mathcal{M}_Q$ are marked by black dots. These are all the points of $\mathcal{M}_Q$, since $\mathcal{D}^1(v) \cap \mathcal{M}_Q = \emptyset$

subcubes contain $\binom{m-2}{2}$ points from $\widetilde{\mathcal{M}}_Q$ in their interior, and the remaining twelve of these triangles contain $m - 2 + \binom{m-2}{2}$ points from $\widetilde{\mathcal{M}}_Q$ in their interior. Hence, it follows that the set $\mathcal{R}^m(v) \cap \widetilde{\mathcal{M}}_Q$ contains exactly $12\ (m-1) + 12\ (m-2) + 24\ \binom{m-2}{2}$ points for $m \in \{2, \dots, q\}$. Therefore, the recurrence relation

$$d_m = d_{m-1} + 24\, m - 36 + 24\, \binom{m-2}{2} \tag{5.3}$$

is satisfied for $m \in \{3, \dots, q\}$. Since $d_2 = 16$, it follows from induction and some elementary computations that $d_q = 4\ (q^3 - 3q^2 + 5q - 2),\ q \geqslant 2$. Since $d_q = \#(\widetilde{\mathcal{M}}_Q)$, the proof of the theorem is complete. $\square$

We proceed by defining another subset $\mathcal{M}_Q \subseteq \mathcal{D}_{q,\Delta_{(1,1,1)}}$, which is also a MDS for $\mathcal{S}_q^1(\triangle_{(1,1,1)}),\ q \geqslant 2$, but different from $\widetilde{\mathcal{M}}_Q$. To do this, again we need some auxiliary sets which we denote by $\Psi^m(v)$, $\Xi^m(v)$, $\Upsilon^m(v)$, and $\Phi^m(v),\ m = 2, \ldots, q$. For $m \in \{2, \ldots, q\}$, we set

$$\Psi^m(v) := \bigcup_{\rho \in \{0,2m\}} \bigcup_{\substack{\sigma \in \{1,\ldots,2m-1\} \\ \tau \in \{\sigma, 2m-\sigma\}}} \bigcup_{\nu=1,2,3} \{\xi^{m,[\pi_\nu(\rho,\sigma,\tau)]}\}$$

and

$$\Xi^m(v) := \{\xi^{m,[2m,2m,2m]}\} \cup \{\xi^{m,[\pi_\nu(2,2m,2m)]},\ \nu = 1, 2, 3\}.$$

The set $\Psi^m(v)$ is similar to the set $\Lambda^m(v)$, but different. The difference is that the points at the eight corners of the subcubes (see previous section) associated with the ring $\mathcal{R}^m(v)$ are not contained in $\Psi^m(v)$. In Figs. 7 and 8 the points from $\Psi^m(v)$ are show as grey dots. Here, we use the above mapping for the rings again (see top of Fig. 3), indicate the essential domain points by grey boxes, and show the cases $q = 3$ (and $m \in \{2, 3\}$) and $q = 2$ (and $m = 2$), respectively. Moreover, the points of $\Xi^m(v)$ are marked as white dots which contain a small black dot. In addition, we let $\Upsilon^2(v) = \emptyset$ and for $m \in \{3, \ldots, q\}$, we set

$$\Upsilon^m(v) := \bigcup_{\substack{\rho \in \{2,\ldots,2(m-1)-2\} \\ \rho \text{ even}}} \bigcup_{\nu=1,2,3} \{\xi^{m,[\pi_\nu(\rho+1,1,2m)]}, \xi^{m,[\pi_\nu(\rho+1,2m,1)]}, \xi^{m,[\pi_\nu(\rho+1,2m,2m-1)]}\}.$$

The set $\Upsilon^m(v)$ is similar to $\Theta^m(v)$, but different. Again, these sets describe certain points being at a distance one to some of the interior triangular faces of $\Delta_{(1,1,1)}$ with two vertices of Q. In Fig. 7, we show the points from $\Upsilon^m(v)$ as white dots—in this case we have $m = 3$, and $\Upsilon^3(v)$ consists of nine points. Moreover, we let for $m \in \{2, \ldots, q-1\}$,

$$\Phi^m(v) := \bigcup_{\substack{\rho \in \{0,\ldots,2m\} \\ \rho \text{ even}}} \bigcup_{\nu=1,2,3} \{\xi^{m,[\pi_\nu(\rho,0,0)]}\},$$

and set $\Phi^q(v) = \emptyset$. The sets $\Phi^m(v)$ describe domain points (outside of $\mathcal{D}^1(v)$) on the interior triangular faces of $\Delta_{(1,1,1)}$ with vertex $(0, 0, 0)$, which do not lie on the boundary of Q. In Fig. 7, we mark the points of the set $\Phi^m(v)$ (where $m = 2$) with a cross. Note that for $q = 2$ there is only one set of the form $\Phi^m(v)$, and this set is empty, while for $q \geqslant 3$ there are $q - 2$ non-empty sets of the form $\Phi^m(v)$.

Roughly speaking, the set $\mathcal{M}_Q$ is now defined by removing the points from the above sets from $\mathcal{D}_{q,\Delta_{(1,1,1)}} \backslash \mathcal{D}^1(v)$. Figs. 7 and 8 illustrate the cases $q = 3$ and 2, respectively. In these figures, $\mathcal{M}_Q$ consists of all the points shown as black dots surrounded by grey boxes. For $q = 3$ the number of these dots is $9 + 43 = 52$, while for $q = 2$ this number is 16. More precisely, we define

$$\mathcal{M}_Q := \bigcup_{m=2}^{q} \Big(\mathcal{R}^m(v) \backslash (\Psi^m(v) \cup \Xi^m(v) \cup \Upsilon^m(v) \cup \Phi^m(v))\Big). \tag{5.4}$$

Theorem 5.3. *The set $\mathcal{M}_Q$ is a minimal determining set for $\mathcal{S}_q^1(\triangle_{(1,1,1)})$, $q \geqslant 2$.*

Proof. It suffices to show that $\mathcal{M}_Q$ is a DS for $\mathcal{S}_q^1(\triangle_{(1,1,1)})$, while the number of points in $\mathcal{M}_Q$ coincides with the dimension of $\mathcal{S}_q^1(\triangle_{(1,1,1)})$, $q \geqslant 2$.

We first show that the cardinality of $\#(\mathcal{M}_Q)$ denoted by c_q coincides with the number given in Theorem 5.2, i.e. we have to show that $c_q = 4\,(q^3 - 3q^2 + 5q - 2),\ q \geqslant 2$. This is certainly true for $q = 2$, since in this case, the set $\mathcal{M}_Q = \mathcal{R}^2(v) \backslash (\Psi^2(v) \cup \Xi^2(v))$ contains 16 points, i.e. $c_2 = 16$. (See Fig. 8, where the points from $\mathcal{M}_Q$ are marked as black dots surrounded by grey boxes.) Moreover, the choice of points in $\mathcal{M}_Q$ implies that for $q \geqslant 3$,

$$\begin{aligned} c_q &= \Big(c_{q-1} - \#(\Phi^{q-1}(v))\Big) + \#\Big(\mathcal{R}^q(v) \cap \mathcal{M}_Q\Big) \\ &= \Big(c_{q-1} - (3\,q - 2)\Big) + \Big((12\,q - 4) - 4 + 15\,(q - 2) + 24\,\tbinom{q-2}{2}\Big) \\ &= c_{q-1} + 24\,q - 36 + 24\,\tbinom{q-2}{2}. \end{aligned}$$

A comparison of this recursion with (5.3) now shows that c_q is the number we claimed.

It remains to show that $\mathcal{M}_Q$ is a DS for $\mathcal{S}_q^1(\triangle_{(1,1,1)}),\ q \geqslant 2$, i.e. we have to show that for any spline $s \in \mathcal{S}_q^1(\triangle_{(1,1,1)})$ with $a_\xi = a_\xi(s) = 0,\ \xi \in \mathcal{M}_Q$, it follows from the C^1-smoothness conditions of the form (4.4), where α, β, and γ are as in (4.6), and (4.7), where $\alpha, \beta, \gamma, \zeta$, and η are as in (4.8), that $s \equiv 0$. We prove this claim by induction on q.

First, we consider the case $q = 2$. Let $s \in \mathcal{S}_2^1(\Delta_{(1,1,1)})$ be given such that $a_\xi = 0$, where $\xi \in \mathcal{M}_Q$, i.e. $\xi \in \mathcal{R}^2(v) \backslash (\Psi^2(v) \cup \Xi^2(v))$. Hence, $a_\xi = 0$, if $\xi = \xi^{2,[\pi_v(0,\sigma,\tau)]}$, where $\sigma, \tau \in \{0, \ldots, 4\}$, σ, τ even, $(\sigma, \tau) \neq (2, 2)$, $v = 1, 2, 3$. It follows from the smoothness conditions of form (4.4), where $\alpha,\ \beta$, and γ are as in (4.6) and $\rho = 0,\ m = 2$, that $a_\xi = 0$ if $\xi = \xi^{2,[\pi_v(0,\sigma,\tau)]},\ \sigma, \tau \in \{1, 3\},\ v = 1, 2, 3$, and $a_\xi = 0$ if $\xi = \xi^{2,[\pi_v(0,2,2)]},\ v = 1, 2, 3$. Moreover, we have $a_\xi(s) = 0$ if $\xi = \xi^{2,[\pi_v(4,1,1)]},\ v = 1, 2, 3$. By using the C^1-smoothness conditions of form (4.7), where $\alpha = \xi^{1,[\pi_v(\sigma,0,0)]},\ \beta,\ \gamma,\ \zeta$, and η are as in (4.8) and $m = 2$, we obtain that $a_\xi = 0,\ \xi \in \mathcal{D}$, where $\mathcal{D}$ is the set defined in (5.1). Therefore, it follows from the arguments given in the beginning of the proof of Theorem 5.1 that $a_\xi = 0,\ \xi \in \mathcal{D}^1(v)$. By using some of the conditions of form (4.7), where $\alpha,\ \beta,\ \gamma,\ \zeta$, and η are as in (4.8) and $m = 2$, we get $a_\xi = 0$, if $\xi = \xi^{2,[\pi_v(1,3,4)]}$ or $\xi = \xi^{2,[\pi_v(1,4,3)]},\ v = 1, 2, 3$. Now, some of the conditions of form (4.4) where $\alpha,\ \beta$, and γ are as in (4.6) and $\rho = 4,\ m = 2$, imply $a_\xi = 0$, if $\xi = \xi^{2,[\pi_v(4,2,2)]},\ v = 1, 2, 3$, and hence, $a_\xi = 0$ if $\xi = \xi^{2,[\pi_v(2,4,4)]}$ or $\xi = \xi^{2,[\pi_v(4,3,3)]},\ v = 1, 2, 3$. It follows from the remaining three C^1-smoothness conditions of form (4.7), i.e. $\alpha = \xi^{1,(2,2,2)}$ in (4.8), that $a_\xi = 0$, if $\xi = \xi^{2,[4,4,4]}$, and hence $s \equiv 0$.

Let us assume that we have already shown that the above claim holds true for $q - 1$, and let a spline $s \in \mathcal{S}_q^1(\triangle_{(1,1,1)}),\ q \geqslant 3$, be given which satisfies $a_\xi(s) = 0,\ \xi \in \mathcal{M}_Q$. Then, it follows from the conditions of form (4.4), where $\alpha,\ \beta$, and γ are as in (4.6) and $\rho = 0,\ m = q$, that $a_\xi(s) = 0$, if $\xi = \xi^{q,[\pi_v(0,\sigma,\tau)]},\ \sigma \in \{1, \ldots, 2q - 1\},\ \tau \in \{\sigma, 2q - \sigma\},\ v = 1, 2, 3$. Moreover, we have $a_\xi = 0$ if $\xi = \xi^{q,[\pi_v(2q,1,1)]},\ v = 1, 2, 3$. By using the C^1-smoothness conditions of form (4.7), where $\alpha = \xi^{m-1,[\pi_v(\sigma,0,0)]},\ \beta,\ \gamma,\ \zeta$, and η are as in (4.8) and

$m = q$, we obtain that $a_\xi = 0,\ \xi \in \Phi^{q-1}(v)$. Therefore, it follows from the choice of points in $\mathcal{M}_Q$ and the induction hypothesis that $a_\xi = 0,\ \xi \in \mathcal{D}^{q-1}(v)$. By using some of the conditions of form (4.7), where $\alpha,\ \beta,\ \gamma,\ \zeta$, and η are as in (4.8) and $m = q$, we get $a_\xi = 0$, if $\xi = \xi^{q,[\pi_v(\rho+1,1,2q)]}$ or $\xi = \xi^{q,[\pi_v(\rho+1,2q,1)]},\ \rho \in \{2, \ldots, 2(q-1)\}$, ρ even, $v = 1, 2, 3$. Note that most of these points are contained in $\Upsilon^q(v)$. Now, three of the conditions of form (4.4) where $\alpha,\ \beta$, and γ are as in (4.6) and $\rho = 2q,\ m = q$, imply $a_\xi = 0$, if $\xi \in \Xi^q(v)\backslash\{\xi^{q,[2q,2q,2q]}\}$. The choice of $\mathcal{M}_Q$ and the C^1-smoothness conditions of form (4.7), where $\alpha = \xi^{q-1,[\pi(\sigma,2q-2,2q-2)]},\ \beta,\ \gamma,\ \zeta$, and η are as in (4.8) and $m = q$, imply that the remaining coefficients associated with points from $\Upsilon^q(v)$ do vanish, i.e. we have $a_\xi = 0$ if $\xi = \xi^{q,[\pi_v(\rho+1,2q,2q-1)]},\ \rho \in \{2, \ldots, 2(q-1)-2\},\ \rho$ even, $v = 1, 2, 3$. Moreover, $a_\xi = 0$ if $\xi = \xi^{q,[\pi_v(2q,2q-1,2q-1)]},\ v = 1, 2, 3$. Therefore, the choice of points in $\mathcal{M}_Q$ and the smoothness conditions of form (4.4) where $\alpha,\ \beta$, and γ are as in (4.6) and $\rho = 2q,\ m = q$, imply $a_\xi = 0$, if $\xi = \xi^{q,[\pi_v(2q,\sigma,\tau)]},\ \sigma \in \{1, \ldots, 2q-1\},\ \tau \in \{\sigma, 2q-\sigma\}$. Moreover, $a_\xi = 0$ if $\xi = \xi^{q,[\pi_v(2q,2q-1,2q-1)]},\ v = 1, 2, 3$. Note that now it can be seen that all the coefficients associated with points from the set $\Psi^q(v)$ vanish. It follows from the remaining three C^1-smoothness conditions of form (4.7), i.e. $\alpha = \xi^{q-1,[2(q-1),2(q-1),2(q-1)]}$ in (4.8), that $a_\xi = 0$, if $\xi = \xi^{q,[2q,2q,2q]}$, and hence $s \equiv 0$.

This completes the proof of the theorem. $\square$

6. A minimal determining set for $\mathcal{S}_q^1(\Delta)$, proof of main results

We construct a MDS $\mathcal{M}$ for $\mathcal{S}_q^1(\triangle),\ q \geqslant 2$, where Δ is a type-6 tetrahedral partition as in Section 3. To do this, we use the results from the previous section. In particular, we use that $\mathcal{M}_Q$ is a MDS for $\mathcal{S}_q^1(\Delta_{(1,1,1)}),\ q \geqslant 2$—a close inspection of the proof of Theorem 3.1 given below shows that this set is in fact needed for three of the four cases which appear in this inductive proof. Counting the number of points in $\mathcal{M}$, we establish the explicit formulae for the dimension given in Theorem 3.1. Note that the construction of $\mathcal{M}$ gives some insight into the structure of the trivariate spline spaces.

In the following, we define $\mathcal{M}$. To do this, we need some auxiliary sets. First, we let $Q = Q_{(1,1,1)}$ again, $\mathcal{M}_Q$ as in (5.4), and set for $i, j, k \in \{1, \ldots, n\}$

$$\mathcal{M}_{(i,j,k)} := \{\xi \in \mathcal{D}_{q,\Delta_{(i,j,k)}} : \xi - \left(\tfrac{i-1}{n}, \tfrac{j-1}{n}, \tfrac{k-1}{n}\right) \in \mathcal{M}_Q\}. \tag{6.1}$$

Hence, $\mathcal{M}_{(i,j,k)}$ is a "shifted" version of the MDS $\mathcal{M}_Q$ for $\mathcal{S}_q^1(\Delta_{(1,1,1)})$ from the previous section. Obviously, we have $\mathcal{M}_{(1,1,1)} = \mathcal{M}_Q$. Moreover, we let $\mathcal{A}_{(i,j,k)},\ i = 2, \ldots, n,\ \ j, k = 1, \ldots, n$, be the set of domain points from $\mathcal{D}_{q,\Delta_{(i,j,k)}}$ which are at a distance zero or one to the left square face $\mathcal{F}^{[1]}_{(i,j,k)}$ of $Q_{(i,j,k)}, \mathcal{B}_{(i,j,k)},\ j = 2, \ldots, n,\ i, k = 1, \ldots, n$, the set of domain points from $\mathcal{D}_{q,\Delta_{(i,j,k)}}$ which are at a distance zero or one to the front square face $\mathcal{F}^{[2]}_{(i,j,k)}$ of $Q_{(i,j,k)}$, and $\mathcal{C}_{(i,j,k)},\ k = 2, \ldots, n,\ i, j = 1, \ldots, n$, the set of domain points from $\mathcal{D}_{q,\Delta_{(i,j,k)}}$ which are at a distance zero or one to the bottom square face $\mathcal{F}^{[3]}_{(i,j,k)}$ of $Q_{(i,j,k)}$. It is not difficult to see that the set $\mathcal{A}_{(i,j,k)}$ contains all the points from $\mathcal{D}_{q,\Delta_{(i,j,k)}}$ where the associated Bernstein–Bézier coefficients are influenced by the

C^1-continuity across the face $\mathcal{F}^{[1]}_{(i,j,k)}$. Similarly, the sets $\mathcal{B}_{(i,j,k)}$ and $\mathcal{C}_{(i,j,k)}$, respectively, contain all the points from $\mathcal{D}_{q,\Delta_{(i,j,k)}}$ where the associated Bernstein–Bézier coefficients are influenced by the C^1-continuity across the faces $\mathcal{F}^{[2]}_{(i,j,k)}$ and $\mathcal{F}^{[3]}_{(i,j,k)}$, respectively. For the case $q = 3$, the sets $\mathcal{A}_{(i,j,k)}$, $\mathcal{B}_{(i,j,k)}$, and $\mathcal{C}_{(i,j,k)}$, are illustrated in Fig. 4, where the points from $\mathcal{A}_{(i,j,k)}$, $\mathcal{B}_{(i,j,k)}$, and $\mathcal{C}_{(i,j,k)}$ are marked as grey dots, large circles and crosses, respectively.

Roughly speaking, the set $\mathcal{M}$ is now defined by choosing the points from the shifted versions $\mathcal{M}_{(i,j,k)}$ of $\mathcal{M}_Q$, and removing those points of these sets which are at a distance zero and one of some of the square faces of $Q_{(i,j,k)}$. Depending on i, j, k there are one, two or three square faces for which the points are removed. More precisely, we define

$$\begin{aligned}
\mathcal{M} := \quad & \mathcal{M}_{(1,1,1)} \\
\cup & \bigcup_{i\in\{2,\ldots,n\}} \Big((\mathcal{M}_{(i,1,1)} \backslash \mathcal{A}_{(i,1,1)}) \cup (\mathcal{M}_{(1,i,1)} \backslash \mathcal{B}_{(1,i,1)}) \\
& \qquad \cup (\mathcal{M}_{(1,1,i)} \backslash \mathcal{C}_{(1,1,i)}) \Big) \\
\cup & \bigcup_{i,j\in\{2,\ldots,n\}} \Big((\mathcal{M}_{(i,j,1)} \backslash (\mathcal{A}_{(i,j,1)} \cup \mathcal{B}_{(i,j,1)})) \cup (\mathcal{M}_{(i,1,j)} \backslash (\mathcal{A}_{(i,1,j)} \\
& \qquad \cup \mathcal{C}_{(i,1,j)})) \cup (\mathcal{M}_{(1,i,j)} \backslash (\mathcal{B}_{(1,i,j)} \cup \mathcal{C}_{(1,i,j)})) \Big) \\
\cup & \bigcup_{i,j,k\in\{2,\ldots,n\}} \Big((\mathcal{M}_{(i,j,k)} \backslash (\mathcal{A}_{(i,j,k)} \cup \mathcal{B}_{(i,j,k)} \cup \mathcal{C}_{(i,j,k)}) \Big).
\end{aligned} \tag{6.2}$$

Theorem 6.1. *The set $\mathcal{M}$ is a minimal determining set for $\mathcal{S}^1_q(\triangle)$, $q \geqslant 2$.*

Proof. Let arbitrary coefficients $a_\xi = a_\xi(s)$, $\xi \in \mathcal{M}$, of a spline $s \in \mathcal{S}^1_q(\triangle)$, $q \geqslant 2$, be given. We have to show that all coefficients of s, i.e. the coefficients a_ξ, where $\xi \in \mathcal{D}_{q,\triangle}$, are uniquely determined, while all the C^1-smoothness conditions of form (4.4), where α, β, and γ are as in (4.5) or (4.6), and (4.7), where $\alpha, \beta, \gamma, \zeta$, and η are as in (4.8), are satisfied.

Our method of proof is to show inductively that the coefficients a_ξ, $\xi \in \mathcal{D}_{q,\triangle_{(i,j,k)}} = \mathcal{D}_{q,\triangle} \cap Q_{(i,j,k)}$ are uniquely determined for $i, j, k \in \{1, \ldots, n\}$, where we consider the cubes $Q_{(i,j,k)}$ in an appropriate order. This natural order is as follows. First, we consider the cases $(i, j, k) = (i, 1, 1)$, $i = 1, \ldots, n$. Then, we consider the cases $(i, j, k) = (1, j, 1)$, $j = 2, \ldots, n$, and $(i, j, k) = (1, 1, k)$, $k = 2, \ldots, n$. Here, we use the result of Theorem 5.3 and we have to take the C^1-continuity across exactly one square face of the cube into account. We proceed by considering the cases $(i, j, k) = (i, j, 1)$, $i, j = 2, \ldots, n$, $(i, j, k) = (i, 1, k)$, $i, k = 2, \ldots, n$, and $(i, j, k) = (1, j, k)$, $j, k = 2, \ldots, n$. Again, we use Theorem 5.3 but now we have to take the C^1-continuity across exactly two square face of the cube (which have a common edge) into account. Finally, we consider the cases (i, j, k), $i, j, k = 2, \ldots, n$. This is the most difficult case. We can use Theorem 5.3 again and have to take the C^1-continuity across exactly three square faces of the cube (which have a common point) into account.

First, it follows from Theorem 5.3 and $\mathcal{M}_{(1,1,1)} = \mathcal{M}_Q \subseteq \mathcal{M}$ that the coefficients a_ξ, where $\xi \in \mathcal{D}_{q,\triangle_{(1,1,1)}} = Q_{(1,1,1)} \cap \mathcal{D}_{q,\triangle}$ are uniquely determined. We proceed by considering the cube $Q_{(2,1,1)}$. This cube has exactly one face in common with $Q_{(1,1,1)}$, namely the face $\mathcal{F}^{[4]}_{(1,1,1)} = \mathcal{F}^{[1]}_{(2,1,1)}$. It follows from the continuity (i.e. (4.2), where α and β are as in (4.3) and $v = 1$), the C^1-smoothness conditions of form (4.4), where α, β, and γ are as in (4.5) and $v = 1$, and the C^1-smoothness conditions of form (4.7), where $\alpha = \xi^{q-1,[0,\rho,\tau]}_{(2,1,1)}$, that the coefficients $a_\xi,\ \xi \in \mathcal{A}_{(2,1,1)}$ are determined. Note that these coefficients are also uniquely determined—this concerns in particular the coefficients a_ξ, where $\xi \in \mathcal{R}^{q-1}(v_{(2,1,1)}) \cap \mathcal{P}^{[1]}_{(2,1,1)}$. By using some elementary computations or an argument similar to Lai and Le Méhauté [16], one can see that if these coefficients are determined using the above conditions, then the C^1-smoothness conditions of form (4.4), where α, β, and γ are as in (4.6) and $\rho = 0,\ v = 1$, are automatically satisfied, too. Here, it is essential that the coefficients a_ξ, where $\xi \in (\mathcal{R}^q(v_{(1,1,1)}) \cup \mathcal{R}^{q-1}(v_{(1,1,1)})) \cap \mathcal{P}^{[4]}_{(1,1,1)}$ already satisfy conditions of this form. The set $\mathcal{M}_Q$ in (5.4) is constructed such that the following property is satisfied: if the coefficients a_ξ, where $\xi \in \mathcal{M}_{(2,1,1)} \cap \mathcal{A}_{(2,1,1)}$, are given, then the coefficients $a_\xi,\ \xi \in \mathcal{A}_{(2,1,1)}$ are uniquely determined from the C^1-smoothness conditions involving these coefficients. Therefore, an argument along the lines of the proof of Theorem 5.3 using the definition of $\mathcal{M}_{(2,1,1)}$ and $\mathcal{M}_{(2,1,1)} \backslash \mathcal{A}_{(2,1,1)} \subseteq \mathcal{M}$ shows that a_ξ is uniquely determined if $\xi \in \mathcal{D}_{q,\triangle_{(2,1,1)}} = Q_{(2,1,1)} \cap \mathcal{D}_{q,\triangle}$, while all the C^1-smoothness conditions of form (4.4) and (4.7), where $i = 2,\ j = 1$, and $k = 1$, are satisfied. It now follows from induction, the choice of points in $\mathcal{M}$, and the same arguments that a_ξ is uniquely determined if $\xi \in \mathcal{D}_{q,\triangle_{(i,1,1)}}$ or $\xi \in \mathcal{D}_{q,\triangle_{(1,i,1)}}$ or $\xi \in \mathcal{D}_{q,\triangle_{(1,1,i)}}$, $i \in \{2, \ldots, n\}$.

Next, we consider the cube $Q_{(2,2,1)}$. According to the above ordering, this cube has exactly two faces in common with some of the cubes considered before, namely the faces $\mathcal{F}^{[4]}_{(1,2,1)} = \mathcal{F}^{[1]}_{(2,2,1)}$ and $\mathcal{F}^{[5]}_{(2,1,1)} = \mathcal{F}^{[2]}_{(2,2,1)}$. It follows from the continuity (i.e. (4.2), where α and β are as in (4.3) and $v \in \{1, 2\}$), the C^1-smoothness conditions of the form (4.4), where α, β, and γ are as in (4.5) and $v \in \{1, 2\}$, and the C^1-smoothness conditions of the form (4.7), where $\alpha = \xi^{q-1,[0,\rho,\tau]}_{(2,2,1)}$ or $\alpha = \xi^{q-1,[\rho,0,\tau]}_{(2,2,1)}$, that the coefficients $a_\xi,\ \xi \in \mathcal{A}_{(2,2,1)} \cup \mathcal{B}_{(2,2,1)}$ are determined. Since the C^1-smoothness conditions along the edge with endpoints $\left(\frac{1}{n}, \frac{1}{n}, 0\right)$ and $\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}\right)$ are consistent, it is clear that the coefficients a_ξ, where $\xi = \xi^{q-1,[0,0,\rho]}_{(2,2,1)},\ \rho \in \{0, \ldots, 2(q-1)\},\ \rho$ even, are uniquely determined. Moreover, using the above argument again, we can see that the coefficients a_ξ, where $\xi \in \mathcal{R}^{q-1}(v_{(2,2,1)}) \cap (\mathcal{P}^{[1]}_{(2,2,1)} \cup \mathcal{P}^{[2]}_{(2,2,1)})$ are uniquely determined while all the C^1-smoothness conditions of form (4.4) involving these coefficients are satisfied. Hence, $a_\xi,\ \xi \in \mathcal{A}_{(2,2,1)} \cup \mathcal{B}_{(2,2,1)}$ are uniquely determined. The set $\mathcal{M}_Q$ in (5.4) is constructed such that the following property is satisfied: if the coefficients a_ξ, where $\xi \in \mathcal{M}_{(2,2,1)} \cap (\mathcal{A}_{(2,2,1)} \cup \mathcal{B}_{(2,2,1)})$, are given, then the coefficients $a_\xi,\ \xi \in \mathcal{A}_{(2,2,1)} \cup \mathcal{B}_{(2,2,1)}$ are uniquely determined from the C^1-smoothness conditions involving these coefficients. Therefore, an argument along the lines of the proof of Theorem 5.3 using the definition of $\mathcal{M}_{(2,2,1)}$ and $\mathcal{M}_{(2,2,1)} \backslash (\mathcal{A}_{(2,2,1)} \cup \mathcal{B}_{(2,2,1)}) \subseteq \mathcal{M}$ shows that a_ξ is uniquely determined if $\xi \in \mathcal{D}_{q,\triangle_{(2,2,1)}} = Q_{(2,2,1)} \cap \mathcal{D}_{q,\triangle}$, while all the C^1-smoothness conditions of form (4.4) and (4.7), where $i = 2,\ j = 2$, and $k = 1$, are satisfied. It now follows from induction, the choice of points in $\mathcal{M}$, and the same arguments that a_ξ is uniquely determined if $\xi \in \mathcal{D}_{q,\triangle_{(i,j,1)}}$ or $\xi \in \mathcal{D}_{q,\triangle_{(i,1,j)}}$ or $\xi \in \mathcal{D}_{q,\triangle_{(1,i,j)}}$, $i, j \in \{2, \ldots, n\}$.

Finally, we consider the cube $Q_{(2,2,2)}$. According to the above ordering, this cube has exactly three faces in common with some of the cubes considered before, namely the faces $\mathcal{F}^{[4]}_{(1,2,2)} = \mathcal{F}^{[1]}_{(2,2,2)}$, $\mathcal{F}^{[5]}_{(2,1,2)} = \mathcal{F}^{[2]}_{(2,2,2)}$, and $\mathcal{F}^{[6]}_{(2,2,1)} = \mathcal{F}^{[3]}_{(2,2,2)}$. It follows from the continuity (i.e. (4.2), where α and β are as in (4.3) and $v \in \{1, 2, 3\}$), the C^1-smoothness conditions of form (4.4), where α, β, and γ are as in (4.5) and $v \in \{1, 2, 3\}$, and the C^1-smoothness conditions of form (4.7), where $\alpha = \xi^{q-1,[\rho,\sigma,\tau]}_{(2,2,2)}$ with $\rho = 0$ or $\sigma = 0$ or $\tau = 0$, that the coefficients $a_\xi,\ \xi \in \mathcal{A}_{(2,2,2)} \cup \mathcal{B}_{(2,2,2)} \cup \mathcal{C}_{(2,2,2)}$ are determined. Since the C^1-smoothness conditions along the edges with endpoints $\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}\right)$ and $\left(\frac{2}{n}, \frac{1}{n}, \frac{1}{n}\right)$, $\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}\right)$ and $\left(\frac{1}{n}, \frac{2}{n}, \frac{1}{n}\right)$, $\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}\right)$ and $\left(\frac{1}{n}, \frac{1}{n}, \frac{2}{n}\right)$, respectively, are pairwise consistent, it is clear that the coefficients a_ξ, where $\xi = \xi^{q-1,[\pi_v(\rho,0,0)]}_{(2,2,2)},\ \rho \in \{0, \ldots, 2(q-1)\},\ \rho$ even, $v = 1, 2, 3$, are uniquely determined. Moreover, using the above argument again, we can see that the coefficients a_ξ, where $\xi \in \mathcal{R}^{q-1}(v_{(2,2,2)}) \cap (\mathcal{P}^{[1]}_{(2,2,2)} \cup \mathcal{P}^{[2]}_{(2,2,2)} \cup \mathcal{P}^{[3]}_{(2,2,2)})$ are uniquely determined while all the C^1-smoothness conditions of form (4.4) involving these coefficients are satisfied. Hence, $a_\xi,\ \xi \in \mathcal{A}_{(2,2,2)} \cup \mathcal{B}_{(2,2,2)} \cup \mathcal{C}_{(2,2,2)}$ are uniquely determined. The set $\mathcal{M}_Q$ in (5.4) is constructed such that the following property is satisfied: if the coefficients a_ξ, where $\xi \in \mathcal{M}_{(2,2,2)} \cap (\mathcal{A}_{(2,2,2)} \cup \mathcal{B}_{(2,2,2)} \cup \mathcal{C}_{(2,2,2)})$, are given, then the coefficients $a_\xi,\ \xi \in \mathcal{A}_{(2,2,2)} \cup \mathcal{B}_{(2,2,2)} \cup \mathcal{C}_{(2,2,2)}$ are uniquely determined from the C^1-smoothness conditions involving these coefficients. Therefore, an argument along the lines of the proof of Theorem 5.3 using the definition of $\mathcal{M}_{(2,2,2)}$ and $\mathcal{M}_{(2,2,2)} \setminus (\mathcal{A}_{(2,2,2)} \cup \mathcal{B}_{(2,2,2)} \cup \mathcal{C}_{(2,2,2)}) \subseteq \mathcal{M}$ shows that a_ξ is uniquely determined if $\xi \in \mathcal{D}_{q,\triangle_{(2,2,2)}} = Q_{(2,2,2)} \cap \mathcal{D}_{q,\triangle}$, while all the C^1-smoothness conditions of form (4.4) and (4.7), where $i = 2,\ j = 2$, and $k = 2$, are satisfied. It now follows from induction, the choice of points in $\mathcal{M}$, and the same arguments that a_ξ is uniquely determined if $\xi \in \mathcal{D}_{q,\triangle_{(i,j,k)}},\ i, j, k \in \{2, \ldots, n\}$.

This shows that all coefficients of s are uniquely determined, while all C^1-smoothness conditions of $\mathcal{S}^1_q(\triangle),\ q \geqslant 2$ are satisfied, and the proof of the theorem is complete. □

By counting the number of points in $\mathcal{M}$, we now obtain the result stated in Theorem 3.1.

Proof of Theorem 3.1. Theorem 5.2 shows that the set $\mathcal{M}_{(1,1,1)} = \mathcal{M}_Q$ contains $4\,(q^3 - 3q^2 + 5q - 2)$ points for $q \geqslant 2$, and it is obvious that this is also the number of points in every set $\mathcal{M}_{(i,j,k)}$ defined in (6.1). Since the cardinality of $\mathcal{M}_{(2,1,1)} \cap \mathcal{A}_{(2,1,1)}$ is $4\,(q^2 - 2q + 3)$, if $q \geqslant 3$, and 11, if $q = 2$, it follows that the set $\mathcal{M}_{(2,1,1)} \setminus \mathcal{A}_{(2,1,1)}$ contains $4\,(q^3 - 4q^2 + 7q - 5)$ points from $\mathcal{M}$, if $q \geqslant 3$, and 5 points from $\mathcal{M}$, if $q = 2$. The same number of points from $\mathcal{M}$ are contained in the cubes $Q_{(i,1,1)}$, $Q_{(1,i,1)}$, and $Q_{(1,1,i)}$ $i = 2, \ldots, n$. Therefore, the total number of points in $\mathcal{M}$ contributed by all of these cubes is $4\,(q^3 - 3q^2 + 5q - 2) + 12\,(n-1)(q^3 - 4q^2 + 7q - 5)$, if $q \geqslant 3$, and $16 + 15\,(n-1)$, if $q = 2$. The cardinality of $\mathcal{M}_{(2,2,1)} \cap (\mathcal{A}_{(2,2,1)} \cup \mathcal{B}_{(2,2,1)})$ is $8q^2 - 19q + 23$, if $q \geqslant 3$, and 15, if $q = 2$. Therefore, the set $\mathcal{M}_{(2,2,1)} \setminus (\mathcal{A}_{(2,2,1)} \cup \mathcal{B}_{(2,2,1)})$ contains $4\,(q^3 - 3q^2 + 5q - 2) - 8q^2 + 19q - 23 = 4q^3 - 20q^2 + 39q - 31$ points from $\mathcal{M}$, if $q \geqslant 3$, and one point from $\mathcal{M}$, if $q = 2$. The same number of points $\mathcal{M}$ are contained in the cubes $Q_{(i,j,1)}$, $Q_{(i,1,j)}$, and $Q_{(1,i,j)},\ i, j = 2, \ldots, n$. Therefore, the total number of points in $\mathcal{M}$ contributed by all of these cubes is $3\,(n-1)^2(4q^3-20q^2+39q-31)$, if $q \geqslant 3$, and $3\,(n-1)^2$, if $q=2$. The cardinality of $\mathcal{M}_{(2,2,2)} \cap (\mathcal{A}_{(2,2,2)} \cup \mathcal{B}_{(2,2,2)} \cup \mathcal{C}_{(2,2,2)})$ is $12q^2-33q+37$, if $q \geqslant 3$, and 16, if $q=2$. Therefore, the set $\mathcal{M}_{(2,2,2)} \backslash (\mathcal{A}_{(2,2,2)} \cup \mathcal{B}_{(2,2,2)} \cup \mathcal{C}_{(2,2,2)})$ contains $4\,(q^3-3q^2+5q-2)-12q^2+33q-37 = 4q^3-24q^2+53q-45$ points from $\mathcal{M}$, if $q \geqslant 3$, and no point from $\mathcal{M}$, if $q=2$. The same number of points $\mathcal{M}$ are contained in the cubes $Q_{(i,j,k)},\ i,j,k=2,\ldots,n$. Therefore, the total number of points in $\mathcal{M}$ contributed by all of these cubes is $(n-1)^3(4q^3-24q^2+53q-45)$, if $q \geqslant 3$, and no point, if $q=2$. Adding these numbers together, an elementary computation now shows that the total number of points in $\mathcal{M}$ coincide with the numbers given in (3.3) and (3.4), respectively.

The proof of Theorem 3.1 is complete. $\square$

7. Remarks

Remark 7.1. The results of this paper can be extended to more general domains where the inductive arguments from the proof of Theorem 6.1 can be applied (see Fig. 9). Simple examples of more general domains are obtained from cube partitions where there are n_j cubes in the jth space direction, $j=1,2,3$, i.e. a total number of $n_1n_2n_3$ cubes. In this case, the dimension of the corresponding spline spaces $\mathcal{S}_q^1(\triangle)$ is given by

$$n_1n_2+n_1n_3+n_2n_3+3\,(n_1+n_2+n_3)+4, \qquad \text{if } q=2$$

and

$$(4q^3-24q^2+53q-45)\,n_1n_2n_3+2\,(2q^2-7q+7)\,(n_1n_2+n_1n_3+n_2n_3) \\ +3\,(q-1)\,(n_1+n_2+n_3)+4, \qquad \text{if } q \geqslant 3.$$

Remark 7.2. In Alfeld et al. [4, Theorem 4] a formula for the dimension for C^1-splines of degree $\geqslant 8$ on generic tetrahedral partitions was given. The numbers given in Theorem 3.1 and Corollary 3.2 do not coincide with these dimensions and therefore we conclude that Δ, the tetrahedral partition defined in Section 3, is non-generic for C^1-splines, in general. Moreover, we note that in Alfeld et al. [4, Examples 7 and 8] as well as in Alfeld et al. [3, Example 26] the dimension of splines on particular cells is computed. These cells are different from the cell considered in Section 5.

Remark 7.3. In Section 3, we compared the dimension of C^1-spline spaces with the dimension of C^0-spline spaces on type-6 tetrahedral partitions, where we observe that the relative difference becomes smaller with larger degrees. We also note that the dimension of trivariate C^1-spline spaces is much larger than that of C^1-tensor spline spaces $S_d^1 \bigotimes S_d^1 \bigotimes S_d^1$ of the same total degree. (Here, S_d^1 is the space of univariate C^1-splines w.r.t. the knots $\frac{i}{n},\ i=0,\ldots,n$.) If $q=3d$, then these spaces are subspaces of $\mathcal{S}_q^1(\triangle)$ which satisfy many super-smoothness conditions across the interior triangular faces of Δ. For instance, it is easy to see that the (tri)quadratic C^1-tensor spline space $S_2^1 \bigotimes S_2^1 \bigotimes S_2^1 \subseteq \mathcal{S}_6^1(\triangle)$, has dimension $n^3+12\,n^2+6\,n+8$, which is much smaller than $273\,n^3+222\,n^2+45\,n+4$, i.e. the dimension of $\mathcal{S}_6^1(\triangle)$. Similarly, the dimension of the subspace $S_3^1 \bigotimes S_3^1 \bigotimes S_3^1 \subseteq \mathcal{S}_9^1(\triangle)$,

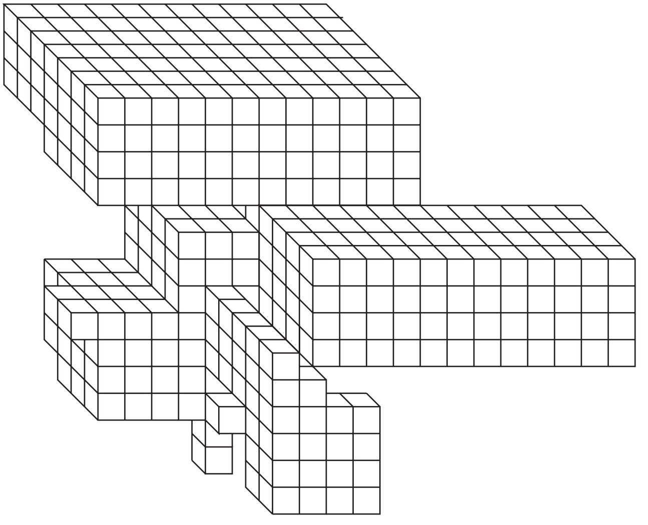

Fig. 9. A more general domain Ω decomposed in uniform cubes

is much smaller than the dimension of $\mathcal{S}_9^1(\triangle)$. Moreover, the local Hermite interpolation approach of Lai and Le Méhauté [16] for type-6 tetrahedral partitions $\varDelta$ is based on a subspace of $\mathcal{S}_5^1(\varDelta)$ of dimension 102 $n^3 + \mathcal{O}(n^2)$. Independently, Schumaker and Sorokina [29] constructed the first box macro element which is based on $\mathcal{S}_6^1(\varDelta)$. This approach uses a subspace of $\mathcal{S}_6^1(\varDelta)$ of dimension 43 $n^3 + \mathcal{O}(n^2)$.

Remark 7.4. The local interpolation methods mentioned in Remark 7.3 yield optimal approximation order $q + 1$ for the spaces $\mathcal{S}_q^1(\triangle)$, if $q \in \{5, 6\}$, and may be generalized to $q \geqslant 7$. However, it is not possible to extend these methods to lower degrees, because of some structural reasons (see, for instance Remark 7.3 in [29]). Currently, only little is known concerning the approximation properties of the spaces when $q \in \{3, 4\}$. Our results presented here indicate that it seems reasonable that for quartic and perhaps for cubic C^1-splines on $\varDelta$, appropriate operators with approximation properties can be defined [30]. This seems not to be possible for the space $\mathcal{S}_2^1(\varDelta)$ which has $\mathcal{O}(n^2)$ degrees of freedom. On the other hand, recently Rössl et al. [23] (see also [18]) applied the structural analysis of this paper to turn it into a practical volume visualization method. This approach uses quadratic C^0-splines on $\varDelta$ satisfying most of the C^1-smoothness properties. The computational results and comparisons presented in these papers showed that from a practical point of view the quadratic splines behave similarly as C^1-functions. The basic idea in this method is to relax some of the conditions of form (4.4) and (4.7) and to replace them by different useful conditions such that appropriate operators for the quadratic splines can be defined which simultaneously approximate the values and the derivatives of smooth trivariate functions. Compared with previously existing methods in the area, the new algorithm combines several advantageous features which are desirable taking in account the specific requirements of efficient volume visualization.

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